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MICRODYNAMICS OF WAVE PROPAGATION

Alberto Puppo, et al

Whittaker Corporation
San Diego, California

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Juan Haener

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Air Force Materials Laboratory
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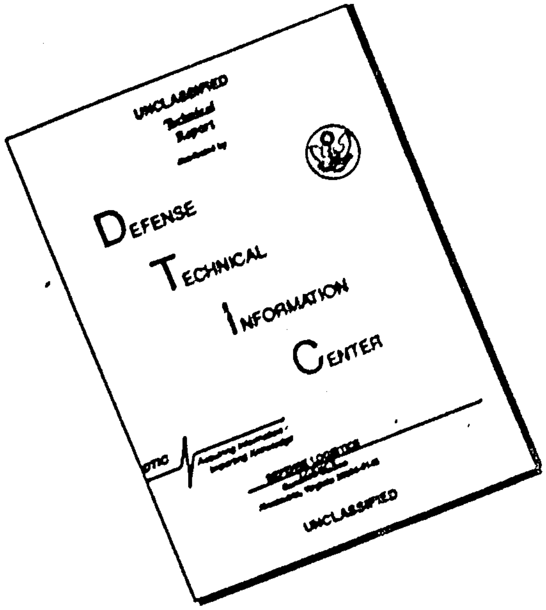

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FOREWORD

This annual summary report was prepared by Whittaker Corporation, Research and Development/San Diego, under Contract F33615-67-C-1894, "Microdynamics of Wave Propagation." Work was accomplished under the direction of Dr. N. J. Pagano, MANC, Nonmetallic Materials Division, Air Force Materials Laboratory, Wright-Patterson Air Force Base, Ohio 45433.

This report covers the period from June 1967 through June 1968. It was released by the authors for publication in July 1968.

Work at Whittaker was conducted under the administration of Mr. Boris Levenetz, Manager of the Advanced Composites Engineering Department. Mr. Alberto Puppo was the Principal Investigator, working under the technical guidance of Dr. Juan Haener, Chief of Analytical Engineering. Mr. Puppo was assisted in his work by Mr. Ming-yuan Feng.

The authors wish to acknowledge Dr. G. Nowak, who, as consultant to this program, provided invaluable assistance in this work.

This technical report has been reviewed and is approved.

J. D. Ray

J. D. Ray, Acting Chief
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ABSTRACT

Part I of this report covers the problem of free and forced vibration of a unidirectional, multifiber reinforced composite. A theoretical investigation is conducted through the use of the linear theory of elasticity. For this case, the geometrical array of the representative element consists of a circular, inner solid fiber cylinder bounded by and bonded to a circular outer matrix shell. Composites of infinite, finite, and semi-infinite lengths are treated. It is assumed that the deformation is axisymmetrical and that the vibration is longitudinal. Characteristic equations are established which relate circular frequencies to axial wave numbers of three cases of composite length. Solutions are obtained for stresses and displacements of composites, of finite or semi-infinite length, subjected to axial, piecewise-constant, or sinusoidal loading at one end and different geometrical boundary conditions at the other.
Part II presents an approximate differential equation based on the Bernoulli hypothesis of deformation. The solution of this equation is established for steady and transient states of vibration in composites of both finite and infinite length. Computation of the coefficients in the differential equation is performed by assuming symmetry of revolution for the basic element and also by using a hexagonal fiber arrangement. Part III lists numerical results based on the equations developed in Parts I and II. The appendixes in this report give the computer programs used to perform the computations.

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A_i, B_i, C_i, D_i $\bar{A}_i, \bar{B}_i, \bar{C}_i, \bar{D}_i$	arbitrary or integration constants when subscripts are used ($i = 1...8$)
A_n, B_n, C_n, D_n	constants
a	fiber radius as defined in (161), or in (173), or in (186) or in (199)
b	matrix radius as defined in (161) or in (199)
c	defined in equation (173) or in (185)
c_1	velocity of dilatation wave in an infinite medium
c_2	velocity of distortion wave in an infinite medium
c_α	phase velocity or propagation velocity
$\lvert d_{ij} \rvert$, $\lvert \bar{d}_{ij} \rvert$	6 X 6 determinants defined in Appendixes III and IV, respectively
E	Young's modulus
e	dilatation defined by $\epsilon_{11} + \epsilon_{22} + \epsilon_{33}$
e_{ij}	Cauchy strain tensor
F_p	Fourier coefficients in equation (79)
G	shear modulus, $= \frac{E}{[2(1+\nu)]}$, or Lamé constant
g^{ij}	associated metric tensor ($i,j = 1,2,3$)
g_{ij}	Euclidean metric tensor ($i,j = 1,2,3$)
I	impact momentum as defined in (144)
I_0, I_1	modified Bessel functions of the first kind, of order zero and one, respectively
J_0, J_1	Bessel functions of the first kind, of order zero and one, respectively
K	constant defined in equation (109)
K_0, K_1	modified Bessel functions of the second kind, of order zero and one, respectively

LIST OF SYMBOLS (Continued)

k	constant defined as +1 whenever it is associated with J,Y, and -1 whenever it is associated with I,K
L	composite length
L_0, L_1	Lamé-Helmholtz displacement potentials ($i = 1,2,3$)
M	number of layers
$M_{i\alpha\beta}$	constants defined by equations (257) ($i = 1...5$)
$M_{i\alpha\beta}$	constants defined by equations (276) ($i = 6...10$)
$\bar{M}_{i\alpha\beta}$	constants defined by equations (291) ($i = 1...5$)
$\|N_{1ij}\|$	5 x 5 determinant defined in Appendix V
$\|N_{2ij}\|$	5 x 5 determinant defined in Appendix VI
$\|\bar{N}_{ij}\|$	5 x 5 determinant defined in Appendix VII
n	integer
P_1, P_0	external force
p	Laplace transform exponent
$Q(r)$, $P(r)$	functions of r in equations (79) and (80)
r,θ,z	cylindrical coordinates
T	period
t	time
u_i	displacement potentials ($i = 1,2,3$)
u,v,w	displacements in r,θ,z directions, respectively
W_0, W_1	denote Bessel functions of the second kind, of order zero and one, respectively, when μ's are real; or modified Bessel functions of the second kind, of order zero and one, respectively, when μ's are imaginery
Y_0, Y_1	Bessel functions of the second kind, of order zero and one, respectively

LIST OF SYMBOLS (Continued)

Z_0, Z_1	denote Bessel functions of the first kind, of order zero and one, respectively, when μ's are real; or modified Bessel functions of the first kind, of order zero and one, respectively, when μ's are imaginary
$\alpha_i, \beta_i, \gamma_i, \delta_i$	arbitrary constants $(i = 1,2)$
$\underline{\alpha}, \underline{\beta}, \underline{\bar{\beta}}, \underline{\gamma}, \underline{\delta}$	indicate the numbers different from $\alpha, \beta, \bar{\beta}, \gamma, \delta$, respectively
β	axial wave number
$\bar{\beta}$	equal to $i\beta$
$\lvert\Delta_{1ij}\rvert$	5 X 5 determinants defined in Appendix V
$\lvert\Delta_{2ij}\rvert$	5 X 5 determinants defined in Appendix VI
$\lvert\bar{\Delta}_{ij}\rvert$	5 X 5 determinants defined in Appendix VII
ϵ	impact duration in seconds
ϵ_{ij}	physical components of strain tensor $(i,j = 1,2,3)$
ϵ_{ijk}	permutation tensor $(i,j,k = 1,2,3)$
ζ	defined in (197)
λ	Lamé constant defined as $\dfrac{E\nu}{[(1+\nu)(1-2\nu)]}$ or wave length
$\mu_{1\alpha\beta}, \mu_{2\alpha\beta}$ $\mu_{1\gamma\delta}, \mu_{2\gamma\delta}$	eigenvalues defined by equations (63) through (66) and equation (288)
$\bar{\mu}_{1\alpha\beta}, \bar{\mu}_{2\alpha\beta}$ $\bar{\mu}_{1\gamma\delta}, \bar{\mu}_{2\gamma\delta}$	moduli of eigenvalues $\mu_{1\alpha\beta}, \mu_{2\alpha\beta}, \mu_{1\gamma\delta}, \mu_{2\gamma\delta}$ respectively
$\bar{\mu}_{1\underline{\alpha\beta}}, \bar{\mu}_{2\underline{\alpha\beta}}$ $\bar{\mu}_{1\underline{\gamma\delta}}, \bar{\mu}_{2\underline{\gamma\delta}}$	eigenvalues different from $\mu_{1\alpha\beta}, \mu_{2\alpha\beta}, \mu_{1\gamma\delta}, \mu_{2\gamma\delta}$, respectively
ν	Poisson's ratio, or as defined in the text
ξ	defined in (197)
ρ	mass density of material
σ_{ij}	physical components of stress tensor associated with coordinate directions as indicated by subscripts $(i,j = 1,2,3)$

LIST OF SYMBOLS (Continued)

τ	variable of convolution integral defined in equation (168)
τ_{ij}	stress tensor (i,j = 1,2,3) or as defined in text
χ_1, χ_2, χ_3 $\chi_4, \bar{\chi}_1, \bar{\chi}_2$	orthagonality factors
ω_α, ω_n	circular frequencies
ω_e	external exciting frequencies
Ω_{ij}	physical components of rotation tensor (i,j = 1,2,3)
$\bar{\Omega}_{ij}$	rotation tensor (i,j = 1,2,3)
$\bar{\Omega}_k$	rotation vector (k = 1,2,3)
$\Omega_1, \Omega_2, \Omega_3$	coefficients of the differential equation
∇^2	Laplacian operator in cylindrical coordinates
I	superscribed for fiber material
II	superscribed for matrix material

PART I

ANALYSIS OF FREE AND FORCED VIBRATION OF A UNIDIRECTIONAL MULTIFIBER REINFORCED COMPOSITE USING EQUATIONS OF THE THEORY OF ELASTICITY

INTRODUCTION AND SUMMARY

This portion of the program was concerned with the analytical investigation of a unidirectional, multifiber reinforced composite subjected to longitudinally forced vibration (dynamic loading at one end) and to free vibration. The theory of elasticity was used for the case of axial symmetry. In this report, solutions to Navier's equations of motion are expressed in the scalar and vector wave potentials associated with the names of Helmholtz and Lamé. Double infinite series solutions for the stresses and displacements of fiber and matrix in their general forms then are established from these functions.

Ahmed [1]* studied the axisymmetric plane strain vibrations of a thick-layered orthotropic cylindrical shells subjected to internal and external pressures. In his analysis, the eigenmodes of the composite shell in terms of the eigenmodes of the individual layers were determined. Using linear theory, Armenàkas [2] solved the problem of free vibration of a single composite cylindrical shell of finite length. No numerical solutions were given in his paper, however.

In this report, a hexagonal array of fibers in a matrix was assumed for the sake of convenience. The basic representative element considered was a circular composite cylinder taken from the whole composite. Specifically, it contained a circular inner solid cylinder of one material bounded by and bonded to a circular outer shell of another material. A model of the element so defined was needed for this investigation. Three different cases of composite length, infinite, finite, and semi-infinite, were considered.

For free vibration, a characteristic equation (frequency equation) which expresses the relationships between circular frequencies and axial wave numbers have been found in the form of a 6 x 6 determinant, transcendental equation. The frequency equations for the infinite and finite cylinder are identical, except that in the latter case, the axial wave numbers are determined by imposing boundary conditions at the ends. For a semi-infinite element, the coefficients in the exponents of the exponential functions in the axial direction in the frequency equation must be real and positive in order to have vanishing stresses and displacements at infinity.

For forced vibration, the analysis centers on the problem of a composite of finite or semi-infinite length, under the axial, piecewise-constant

*Numbers in the bracket designate references at the end of the report.

or sinusoidal loading at one end. The boundary geometry at the nonloading end of the finite composite cylinder is either fixed or freely supported. Solutions of stresses and displacements of fiber and matrix for the aforementioned cases have been obtained through the generalized Fourier series technique, which permits one to determine the eigenmodes of the composite element, in terms of the eigenmodes of individual constituents. The concept of quasi-orthogonality was initiated by Tittle and is now used in a rigorous expansion of the boundary functions traversing two regions into a series of nonorthogonal eigensets that arise from the solutions of the potentials in two different media. In other words, the eigenfunctions are not orthogonal over the total interval in the radial direction, because the conditions of the Sturm-Liouville problem are not satisfied. Specifically, the physical constants of the governing differential equations of Lamé-Helmholtz potentials of a composite are different for each constituent. Therefore, it is impossible to represent a function across the boundary as the expansions of such nonorthogonal eigensets in the conventional way; for example, by means of Fourier-Bessel or Dini-Bessel expansion. To this end, orthogonal sets must be constructed from the quasi-orthogonal eigensets by the use of orthogonality factors for each medium from the orthogonality conditions.

In formulating and solving the problem, the following considerations and assumptions prevail:

1. Both materials are elastic, isotropic, and homogeneous.
2. Body forces and dissipative forces are neglected.
3. Density as well as velocities of dilatational and distortional waves in an infinite medium of both constituents are constants.
4. Only small displacements are considered; in other words, squares and products of angles of rotation are negligibly small in comparison with elongations and shears.
5. Deformation is axisymmetrical.
6. The vibration is longitudinal, nontorsional, and non-bending.
7. Dynamic buckling phenomena are not considered.
8. Applied force is independent of deformation.
9. Continuity of displacements and stresses at the fiber-matrix interface is ensured.

GENERAL SOLUTIONS OF DISPLACEMENTS AND STRESSES IN TERMS OF LAMÉ-HELMHOLTZ POTENTIALS

In the absence of prescribed body forces, Navier's equation of motion in linear elasticity for a homogeneous, isotropic medium is, in a general coordinate system,

$$g^{jk}\, u_{i,jk} + \frac{1}{2(1-2\nu)} \left[g^{jk}(u_{j,k} + u_{k,j}) \right]_{,i} = \frac{\rho}{G} \ddot{u}_i \tag{1}$$

where g^{jk} is the associated metric tensor, ν is Poisson's ratio, G is a Lamé constant, ρ is mass density of the material, and repeated indices indicate summation.

In a cylindrical coordinates (r,θ,z) system, equation (1) can be written in the following manner: [15],[17],[27],[28]

$$\nabla^2 u + \frac{1}{1-2\nu}\left(\frac{\partial e}{\partial r} - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right) = \frac{\rho}{G}\frac{\partial^2 u}{\partial t^2}$$

$$\nabla^2 v + \frac{1}{1-2\nu}\left(\frac{1}{r}\frac{\partial e}{\partial \theta} + \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{v}{r^2}\right) = \frac{\rho}{G}\frac{\partial^2 v}{\partial t^2}$$

$$\nabla^2 w + \frac{1}{1-2\nu}\frac{\partial e}{\partial z} = \frac{\rho}{G}\frac{\partial^2 w}{\partial t^2} \qquad (2)$$

where $u = \sqrt{g^{11}}\, u_1$, $v = \sqrt{g^{22}}\, u_2$, $w = \sqrt{g^{33}}\, u_3$ and ∇^2 is the Laplacian operator, defined as

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \qquad (3)$$

and e is the dilation defined by

$$e = \frac{1}{r}\frac{\partial(ru\ )}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} \qquad (4)$$

Equations (2) are often associated with the names of Pochhammer and Chree.

Based on the Helmholtz theorem, Lamé suggested that a general solution to the differential equation (1) assumes the following form: [8],[9],[11],[20],[21],[25]

$$u_i = \sqrt{g^{ii}}\, L_{o,i} + \sqrt{g^{jk}}\, \epsilon_{ijk}\, L_{k,j} \qquad (5)$$

where $i,j,k = 1,2,3$; i is not summed in $\sqrt{g^{ii}}$ and $\sqrt{g^{jk}}$, ϵ_{ijk} is the permutation tensor, and L_o , L_1 (1,2,3) are the displacement potentials, which are called Lame-Helmholtz potentials in this report, such that

$$g^{jk}\, L_{o,jk} = \frac{1}{c_1^2}\frac{\partial^2 L_o}{\partial t^2} \qquad (6)$$

$$g^{jk}\, L_{i,jk} = \frac{1}{c_2^2}\frac{\partial^2 L_i}{\partial t^2} \tag{7}$$

and

$$L_{i,i} = 0 \tag{8}$$

Here

$$c_1 = \left(\frac{2G+\lambda}{\rho}\right)^{\frac{1}{2}} \tag{9}$$

and

$$c_2 = \left(\frac{G}{\rho}\right)^{\frac{1}{2}} \tag{10}$$

are the velocities of dilation and distortion waves, respective, in an infinite medium, and λ is the Lamé constant [20]. Equations (6) and (7) are scalar and vector wave equations, respectively. Written out in scalar form in cylindrical coordinates, equation (5) becomes

$$\begin{aligned}
u &= \frac{\partial L_0}{\partial r} - \frac{\partial L_2}{\partial z} + \frac{1}{r}\frac{\partial L_3}{\partial \theta} \\
v &= \frac{1}{r}\frac{\partial L_0}{\partial \theta} + \frac{\partial L_1}{\partial z} - \frac{\partial L_3}{\partial r} \\
w &= \frac{\partial L_0}{\partial z} - \frac{1}{r}\frac{\partial L_1}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial r}(rL_2)
\end{aligned} \tag{11}$$

The strain tensor is expressed as

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{12}$$

Its corresponding physical components of strain tensor, in general coordinates, are

$$\epsilon_{ij} = \sqrt{g^{ii}}\sqrt{g^{jj}}\; e_{ij} \tag{13}$$

where i,j are not summed. In cylindrical coordinates, the physical components of strain tensor, derived from equations (12) and (13), are

$$\epsilon_{11} = \frac{\partial u}{\partial r}$$

$$\epsilon_{22} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\epsilon_{33} = \frac{\partial w}{\partial z}$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right)$$

$$\epsilon_{13} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)$$

$$\epsilon_{23} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}\right) \tag{14}$$

Substituting equations (11) into equations (14) gives the strain in terms of potentials in cylindrical coordinates:

$$\epsilon_{11} = \frac{\partial^2 L_0}{\partial r^2} - \frac{\partial^2 L_2}{\partial r \partial z} + \frac{1}{r}\frac{\partial^2 L_3}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial L_3}{\partial \theta}$$

$$\epsilon_{22} = \frac{1}{r}\left(\frac{\partial L_0}{\partial r} + \frac{1}{r}\frac{\partial^2 L_0}{\partial \theta^2} + \frac{\partial^2 L_1}{\partial \theta \partial z} - \frac{\partial L_2}{\partial z} + \frac{1}{r}\frac{\partial L_3}{\partial \theta} - \frac{\partial^2 L_3}{\partial r \partial \theta}\right)$$

$$\epsilon_{33} = \frac{\partial^2 L_0}{\partial z^2} - \frac{1}{r}\frac{\partial^2 L_1}{\partial \theta \partial z} + \frac{1}{r}\frac{\partial^2}{\partial r \partial z}(rL_2)$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{2}{r}\frac{\partial^2 L_0}{\partial r \partial \theta} - \frac{2}{r^2}\frac{\partial L_0}{\partial \theta} + \frac{\partial^2 L_1}{\partial r \partial z} - \frac{1}{r}\frac{\partial L_1}{\partial z} - \right.$$

$$\left. - \frac{1}{r}\frac{\partial^2 L_2}{\partial \theta \partial z} + \frac{1}{r}\frac{\partial L_3}{\partial r} - \frac{\partial^2 L_3}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 L_3}{\partial \theta^2}\right)$$

$$\epsilon_{13} = \frac{1}{2}\left(2\frac{\partial^2 L_0}{\partial r \partial z} + \frac{1}{r^2}\frac{\partial L_1}{\partial \theta} - \frac{1}{r}\frac{\partial^2 L_1}{\partial \theta^2} - \frac{L_2}{r^2} + \right.$$

$$\left. + \frac{1}{r}\frac{\partial L_2}{\partial r} + \frac{\partial^2 L_2}{\partial r^2} - \frac{\partial^2 L_2}{\partial z^2} + \frac{1}{r}\frac{\partial^2 L_3}{\partial \theta \partial z}\right)$$

$$\epsilon_{23} = \frac{1}{2}\left(\frac{2}{r}\frac{\partial^2 L_0}{\partial \theta \partial z} - \frac{1}{r^2}\frac{\partial^2 L_1}{\partial \theta^2} + \frac{\partial^2 L_1}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2}{\partial r \partial \theta}(rL_2) - \frac{\partial^2 L_3}{\partial r \partial z}\right) \tag{15}$$

The dilatation e in cylindrical coordinates, then, is

$$e = \frac{\partial^2 L_0}{\partial r^2} + \frac{1}{r}\frac{\partial L_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 L_0}{\partial \theta^2} + \frac{\partial^2 L_0}{\partial z^2} = \nabla^2 L_0 \tag{16}$$

The rotation tensor is

$$\bar{\Omega}_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) \tag{17}$$

Then the rotation vector in general coordinates is

$$\bar{\Omega}_k = \frac{1}{2}\epsilon_{kij}\,\Omega_{ij} \tag{18}$$

where Ω_{ij} are the physical components of rotation tensor defined by

$$\Omega_{ij} = \sqrt{g^{ii}}\sqrt{g^{jj}}\,\bar{\Omega}_{ij} \tag{19}$$

where i,j are not summed.

In cylindrical coordinates, equations (17) through (19) become

$$\bar{\Omega}_1 = -\frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{1}{r}\frac{\partial w}{\partial \theta}\right)$$

$$\bar{\Omega}_2 = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)$$

$$\bar{\Omega}_3 = -\frac{1}{2r}\left(\frac{\partial u}{\partial \theta} - \frac{\partial (rv)}{\partial r}\right) \tag{20}$$

From equations (11) and (20), we have

$$\bar{\Omega}_1 = -\frac{1}{2}\left(\frac{\partial^2 L_1}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 L_1}{\partial \theta^2} - \frac{1}{r^2}\frac{\partial^2}{\partial r \partial \theta}(rL_2) - \frac{\partial^2 L_3}{\partial r \partial z}\right)$$

$$\bar{\Omega}_2 = \frac{1}{2}\left(\frac{1}{r}\frac{\partial^2 L_1}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial L_1}{\partial \theta} + \frac{L_2}{r^2} - \frac{1}{r}\frac{\partial L_2}{\partial r} - \frac{\partial^2 L_2}{\partial r^2} - \frac{\partial^2 L_2}{\partial z^2} + \frac{1}{r}\frac{\partial_2 L_3}{\partial \theta \partial z}\right)$$

$$\bar{\Omega}_3 = -\frac{1}{2r}\left(\frac{\partial L_1}{\partial z} + r\frac{\partial^2 L_1}{\partial r \partial z} + \frac{\partial^2 L_2}{\partial \theta \partial z} - \frac{\partial L_3}{\partial r} - r\frac{\partial^2 L_3}{\partial r^2} - \frac{1}{r}\frac{\partial^2 L_3}{\partial \theta^2}\right) \tag{21}$$

In the case of a homogeneous, isotropic medium, the generalized Hooke's law which relates the physical components of stress tensor to that of the strain tensor in general coordinates assumes the following form:

$$\sigma_{ij} = \lambda g_{ij}\, g^{ij}\, \epsilon_{ij} + 2G\, \epsilon_{ij} \tag{22}$$

where g_{ij} is the Euclidean metric tensor. The relationship between the stress tensor and the Cauchy strain tensor has the same form as that given in equation (22), since

$$\sigma_{ij} = \sqrt{g^{ii}}\sqrt{g^{jj}}\, \tau_{ij} \tag{23}$$

where τ_{ij} is the stress tensor and i, j are not summed. Combining equations (15), (16), and (22), we obtain the stress components, in terms of Lamé-Helmholtz potentials, in cylindrical coordinates as

$$\sigma_{11} = \lambda \nabla^2 L_0 + 2G\left(\frac{\partial^2 L_0}{\partial r^2} - \frac{\partial^2 L_2}{\partial r \partial z} + \frac{1}{r}\frac{\partial^2 L_3}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial L_3}{\partial \theta}\right)$$

$$\sigma_{22} = \lambda\nabla^2 L_0 + \frac{2G}{r}\left(\frac{\partial L_0}{\partial r} + \frac{1}{r}\frac{\partial^2 L_0}{\partial \theta^2} + \frac{\partial^2 L_1}{\partial\theta\partial z} - \frac{\partial L_2}{\partial z} + \frac{1}{r}\frac{\partial L_3}{\partial \theta} - \frac{\partial^2 L_3}{\partial r\partial\theta}\right)$$

$$\sigma_{33} = \lambda\nabla^2 L_0 + 2G\left(\frac{\partial^2 L_0}{\partial z^2} - \frac{1}{r}\frac{\partial^2 L_1}{\partial\theta\partial z} + \frac{1}{r}\frac{\partial^2}{\partial r\partial z}(rL_2)\right)$$

$$\sigma_{12} = G\left(\frac{2}{r}\frac{\partial^2 L_0}{\partial r\partial\theta} - \frac{2}{r^2}\frac{\partial L_0}{\partial\theta} + \frac{\partial^2 L_1}{\partial r\partial z} - \frac{1}{r}\frac{\partial L_1}{\partial z} - \frac{1}{r}\frac{\partial^2 L_2}{\partial\theta\partial z} + \right.$$
$$\left. \frac{1}{r}\frac{\partial L_3}{\partial r} - \frac{\partial^2 L_3}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 L_3}{\partial\theta^2}\right)$$

$$\sigma_{13} = G\left(2\frac{\partial^2 L_0}{\partial r\partial z} + \frac{1}{r^2}\frac{\partial L_1}{\partial\theta} - \frac{1}{r}\frac{\partial^2 L_1}{\partial\theta^2} - \frac{L_2}{r^2} + \frac{1}{r}\frac{\partial L_2}{\partial r} + \right.$$
$$\left. \frac{\partial^2 L_2}{\partial r^2} - \frac{\partial^2 L_2}{\partial z^2} + \frac{1}{r}\frac{\partial^2 L_3}{\partial\theta\partial z}\right)$$

$$\sigma_{23} = G\left(\frac{2}{r}\frac{\partial^2 L_0}{\partial\theta\partial z} - \frac{1}{r^2}\frac{\partial^2 L_1}{\partial\theta^2} + \frac{\partial^2 L_1}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2}{\partial r\partial\theta}(rL_2) - \frac{\partial^2 L_3}{\partial r\partial z}\right) \qquad (24)$$

In this analysis, the hexagonal array of fibers is assumed. A basic, representative element, which is a composite cylinder taken from a composite of infinite size, contains a continuous, circular inner solid cylinder of fiber bounded by and bonded to an outer shell of matrix, the contour of which is approximated by a circle. The geometry and coordinates system for an elemental composite cylinder are depicted in Figure 1.

SOLUTIONS OF POTENTIALS IN THE CASE OF AXIALLY SYMMETRIC DEFORMATION AND LONGITUDINAL VIBRATION

In the case of axially symmetric deformation and longitudinal vibration, we have

$$v = \sigma_{12} = \sigma_{23} = \bar{\Omega}_1 = \bar{\Omega}_3 = 0 \qquad (25)$$

Therefore, when written out in scalar form in cylindrical coordinates, equations (6) and (7) become

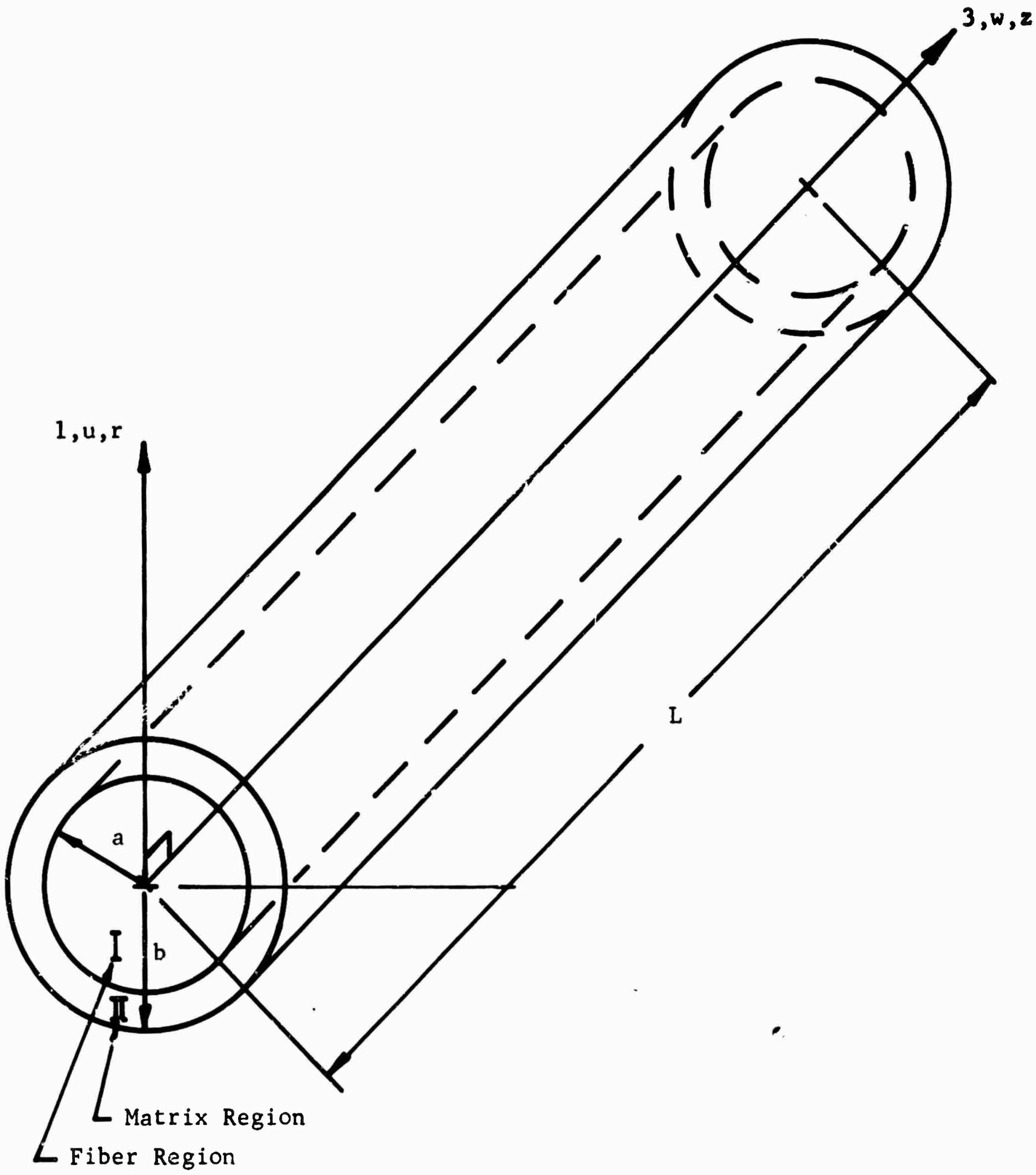

Figure 1. Geometry and Coordinates System for a Basic Representative Element Composite of Finite Length. For a composite of semi-infinite length, $L = \infty$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)L_0 = 0 \tag{26}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}\right)L_2 = 0 \tag{27}$$

The foregoing equations can be solved by the method of separation of the variables.

Omitting the routing procedure, we arrive at the general product solutions of equations (26) and (27) as follows.

For the case of infinite and finite length,

$$\begin{aligned}
L_0 = \sum_{\alpha_1 \geq 0}^{\infty} \Bigg\{ \sum_{\beta_1 \geq 0}^{\infty} \Big[& A_{1\alpha\beta} \sin(\beta_1 z) \sin(\alpha_1 c_1^{II} t) + A_{3\alpha\beta} \sin(\beta_1 z) \cos(\alpha_1 c_1^{II} t) + \\
& A_{5\alpha\beta} \cos(\beta_1 z) \sin(\alpha_1 c_1^{II} t) + A_{7\alpha\beta} \cos(\beta_1 z) \cos(\alpha_1 c_1^{II} t)\Big] \; Z_0\left(\bar{\mu}_{1\alpha\beta} r\right) + \\
\Big[& A_{2\alpha\beta} \sin(\beta_1 z) \sin(\alpha_1 c_1^{II} t) + A_{4\alpha\beta} \sin(\beta_1 z) \cos(\alpha_1 c_1^{II} t) + \\
& A_{6\alpha\beta} \cos(\beta_1 z) \sin(\alpha_1 c_1^{II} t) + A_{8\alpha\beta} \cos(\beta_1 z) \cos(\alpha_1 c_1^{II} t)\Big] \; W_0\left(\bar{\mu}_{1\alpha\beta} r\right)\Bigg\} + \\
\sum_{\bar{\mu}_{1\alpha} = \alpha_1 > 0}^{\infty} \Bigg\{ \Big[& A_{1\alpha} z \sin\left(\bar{\mu}_{1\alpha} c_1^{II} t\right) + A_{3\alpha} z \cos\left(\bar{\mu}_{1\alpha} c_1^{II} t\right)\Big] Z_0\left(\bar{\mu}_{1\alpha} r\right) + \\
\Big[& A_{2\alpha} z \sin\left(\bar{\mu}_{1\alpha} c_1^{II} t\right) + A_{4\alpha} z \cos\left(\bar{\mu}_{1\alpha} c_1^{II} t\right)\Big] W_0\left(\bar{\mu}_{1\alpha} r\right)\Bigg\} + \\
& A_{10} z + A_{20}(\log r) z + A_{50} + A_{80} \log r
\end{aligned} \tag{28}$$

where $\bar{\mu}_{1\alpha\beta}$ and $\bar{\mu}_{1\alpha}$ are moduli of $\mu_{1\alpha\beta}$ and $\mu_{1\alpha}$, respectively, and

$$\mu_{1\alpha\beta}^2 = \alpha_1^2 - \beta_1^2 \quad , \quad \mu_{1\alpha} = \alpha_1 \tag{29}$$

and Z_o and W_o denote Bessel functions $J_o\left(\mu_{1\alpha\beta}\right)$ and $Y_o\left(\mu_{1\alpha\beta}\right)$ when $\mu_{1\alpha\beta}$ is real, or modified Bessel functions $I_o\left(\bar{\mu}_{1\alpha\beta}r\right)$ and $K_o\left(\bar{\mu}_{1\alpha\beta}r\right)$, respectively, when $\mu_{1\alpha\beta}$ is imaginary, and α_1, β_1, are eigenvalues which depend upon the boundary conditions in a given problem. In addition,

$$I_2 = \sum_{\alpha_2 \geq 0}^{\infty} \left\{ \sum_{\beta_2 \geq 0}^{\infty} \left(\left[B_{1\alpha\beta} \sin(\beta_2 z) \sin(\alpha_2 c_2^{II} t) + B_{3\alpha\beta} \sin(\beta_2 z) \cos(\alpha_2 c_2^{II} t) + \right.\right.\right.$$

$$\left. B_{5\alpha\beta} \cos(\beta_2 z) \sin(\alpha_2 c_2^{II} t) + B_{7\alpha\beta} \cos(\beta_2 z) \cos(\alpha_2 c_2^{II} t) \right] Z_1\left(\mu_{2\alpha\beta} r\right) +$$

$$\left[B_{2\alpha\beta} \sin(\beta_2 z) \sin(\alpha_2 c_2^{II} t) + B_{4\alpha\beta} \sin(\beta_2 z) \cos(\alpha_2 c_2^{II} t) + \right.$$

$$\left.\left.\left. B_{6\alpha\beta} \cos(\beta_2 z) \sin(\alpha_2 c_2^{II} t) + B_{8\alpha\beta} \cos(\beta_2 z) \cos(\alpha_2 c_2^{II} t) \right] W_1\left(\bar{\mu}_{2\alpha\beta} r\right) \right) \right\} +$$

$$\sum_{\bar{\mu}_{2\alpha} = \alpha_2 > 0}^{\infty} \left\{ \left[B_{1\alpha} z \sin\left(\bar{\mu}_{2\alpha} c_2^{II} t\right) + B_{3\alpha} z \cos\left(\bar{\mu}_{2\alpha} c_2^{II} t\right) \right] Z_1\left(\bar{\mu}_{2\alpha} r\right) + \right.$$

$$\left. \left[B_{2\alpha} z \sin\left(\bar{\mu}_{2\alpha} c_2^{II} t\right) + B_{4\alpha} z \cos\left(\bar{\mu}_{2\alpha} c_2^{II} t\right) \right] W_1\left(\bar{\mu}_{2\alpha} r\right) \right\} +$$

$$B_{10}\, rz + B_{20}\, r^{-1} z + B_{50}\, r + B_{60}\, r^{-1} \tag{30}$$

where $\bar{\mu}_{2\alpha\beta}$ and $\bar{\mu}_{2\alpha}$ are the moduli of $\mu_{2\alpha\beta}$ and $\mu_{2\alpha}$, respectively, and

$$\mu_{2\alpha\beta}^2 = \alpha_2^2 - \beta_2^2 \quad , \quad \mu_{2\alpha} = \alpha_2 \tag{31}$$

and Z_1 and W_1 denote Bessel functions $J_1\left(\mu_{2\alpha\beta}\right)$ and $Y_1\left(\mu_{2\alpha\beta} r\right)$ respectively when $\mu_{2\alpha\beta}$ is real, or modified Bessel functions $I_1\left(\bar{\mu}_{2\alpha\beta} r\right)$ and $K_1\left(\bar{\mu}_{2\alpha\beta} r\right)$ respectively, when $\mu_{2\alpha\beta}$ is imaginary.

In case of modes exited below their cut-off frequency, attenuated waves exist which may be described by the following solution

For the case of semi-infinite length,

$$
\begin{aligned}
L_o = \sum_{\alpha_1 \geq 0}^{\infty} \Bigg\{ \sum_{\bar{\beta}_1 \geq 0}^{\infty} \Bigg(& \left[\bar{A}_{5\alpha\beta}\, e^{-\bar{\beta}_1 z} \sin(\alpha_1 c_1^{II} t) + \bar{A}_{7\alpha\beta}\, e^{-\bar{\beta}_1 z} \cos(\alpha_1 c_1^{II} t) \right] \cdot \\
& J_o\left(\mu_{1\alpha\beta} r\right) + \left[\bar{A}_{6\alpha\beta}\, e^{-\bar{\beta}_1 z} \sin(\alpha_1 c_1^{II} t) + \bar{A}_{8\alpha\beta}\, e^{-\bar{\beta}_1 z} \cos(\alpha_1 c_1^{II} t) \right] \cdot \\
& Y_o\left(\mu_{1\alpha\beta} r\right) \Bigg) \Bigg\} + \sum_{\mu_{1\alpha} = \alpha_1 > 0}^{\infty} \Bigg\{ \left[\bar{A}_{5\alpha} z \sin\left(\mu_{1\alpha} c_1^{II} t\right) + \bar{A}_{7\alpha} z \cos\left(\mu_{1\alpha} c_1^{II} t\right) \right] \cdot \\
& J_o\left(\mu_{1\alpha} r\right) + \left[\bar{A}_{6\alpha} z \sin\left(\mu_{1\alpha} c_1^{II} t\right) + \bar{A}_{8\alpha} z \cos\left(\mu_{1\alpha} c_1^{II} t\right) \right] Y_o\left(\mu_{1\alpha} r\right) \Bigg\} + \\
& \bar{A}_{10} z + \bar{A}_{20} (\log r)\, z + \bar{A}_{50} + \bar{A}_{60} (\log r)
\end{aligned}
\tag{32}
$$

and

$$
\begin{aligned}
L_2 = \sum_{\alpha_2 \geq 0}^{\infty} \Bigg\{ \sum_{\bar{\beta}_2 \geq 0}^{\infty} \Bigg(& \left[\bar{B}_{5\alpha\beta}\, e^{-\bar{\beta}_2 z} \sin(\alpha_2 c_2^{II} t) + \bar{B}_{7\alpha\beta}\, e^{-\bar{\beta}_2 z} \cos(\alpha_2 c_2^{II} t) \right] \cdot \\
& J_1\left(\mu_{2\alpha\beta} r\right) + \left[\bar{B}_{6\alpha\beta}\, e^{-\bar{\beta}_2 z} \sin(\alpha_2 c_2^{II} t) + \bar{B}_{8\alpha\beta}\, e^{-\bar{\beta}_2 z} \cos(\alpha_2 c_2^{II} t) \right] \cdot \\
& Y_1\left(\mu_{2\alpha\beta} r\right) \Bigg) \Bigg\} + \sum_{\mu_{2\alpha} = \alpha > 0}^{\infty} \Bigg\{ \left[\bar{B}_{5\alpha} z \sin\left(\mu_{2\alpha} c_2^{II} t\right) + \bar{B}_{7\alpha} z \cos\left(\mu_{2\alpha} c_2^{II} t\right) \right] \cdot \\
& J_1\left(\mu_{2\alpha} r\right) + \left[\bar{B}_{6\alpha} z \sin\left(\mu_{2\alpha} c_2^{II} t\right) + \bar{B}_{8\alpha} z \cos\left(\mu_{2\alpha} c_2^{II} t\right) \right] Y_1\left(\mu_{2\alpha} r\right) \Bigg\} + \\
& \bar{B}_{10} r z + \bar{B}_{20} r^{-1} z + \bar{B}_{50} r + \bar{B}_{60} r^{-1}
\end{aligned}
\tag{33}
$$

where, in equations (32) and (33),

$$\mu_{1\alpha\beta}^2 = \alpha_1^2 + \overline{\beta}_1^2 \quad , \quad \mu_{1\alpha} = \alpha_1 \tag{34}$$

and

$$\mu_{2\alpha\beta}^2 = \alpha_2^2 + \overline{\beta}_2^2 \quad , \quad \mu_{2\alpha} = \alpha_2 \tag{35}$$

In equations (28) through (35) the expressions are for the matrix. For the fiber, we must:

1. Replace $A_{i\alpha\beta}$ by $C_{i\gamma\delta}$ $(i = 1,3,5,7)$

   $\overline{A}_{i\alpha\beta}$ by $\overline{C}_{i\gamma\delta}$ $(i = 5,7)$

   $B_{i\alpha\beta}$ by $D_{i\gamma\delta}$ $(i = 1,3,5,7)$

   $\overline{B}_{i\alpha\beta}$ by $\overline{D}_{i\gamma\delta}$ $(i = 5,7)$ (36)

2. Then let $A_{i\alpha\beta}$, $B_{i\alpha\beta}$, $\overline{A}_{i\alpha\beta}$, $\overline{B}_{i\alpha\beta}$ $(i = 2,4,6,8) = 0$ (37)
   so that we will have finite values of stresses and displacements at $r = 0$

3. Replace α_1 , β_1 , $\mu_{1\alpha\beta}$, $\mu_{1\alpha}$, α_2 , β_2 , $\mu_{2\alpha\beta}$, $\mu_{2\alpha}$, $\overline{\beta}_1$, $\overline{\beta}_2$

   with γ_1 , δ_1 , $\mu_{1\gamma\delta}$, $\mu_{1\gamma}$, γ_2 , δ_2 , $\mu_{2\gamma\delta}$, $\mu_{2\gamma}$, $\overline{\delta}_1$, $\overline{\delta}_2$ (38)

4. Replace A_{10} , A_{50} , B_{10} , B_{50} , $\overline{A}_{10}$, $\overline{A}_{50}$, $\overline{B}_{10}$, $\overline{B}_{50}$

   by C_{10} , C_{50} , D_{10} , D_{50} , $\overline{C}_{10}$, $\overline{C}_{50}$, $\overline{D}_{10}$, $\overline{D}_{50}$,

   and set A_{20} , A_{60} , B_{20} , B_{60} , $\overline{A}_{20}$, $\overline{A}_{60}$, $\overline{B}_{20}$, $\overline{B}_{60}$

   to zero.

5. Replace c_1^{II}, c_2^{II} by c_1^{I} , c_2^{I} , respectively for the reinforcement

In equations (28) and (30), the finiteness of potentials, which in turn are the finiteness of displacements and stresses, has been satisfied as t approaches infinity.

DOUBLE INFINITE SERIES SOLUTIONS OF DISPLACEMENTS AND STRESSES (GENERAL FORM)

The solutions of displacements and stresses in the form of double infinite series are obtainable by substituting equations (28) and (30), or equations **(32) and (33)**, into equations (11) and (24). The expressions obtained for the cases of infinite and finite length cylinders as well as for semi-infinite length cylinders, are written out in Appendixes I and II, respectively. These equations are the general solutions of the matrix. For the fiber, the results are of the same form, but $A_{i\alpha\beta}$, $B_{i\alpha\beta}$, $\overline{A}_{i\alpha\beta}$, $\overline{B}_{i\alpha\beta}$ are replaced with $C_{i\alpha\beta}$, $D_{i\alpha\beta}$, $\overline{C}_{i\alpha\beta}$, and $\overline{D}_{i\alpha\beta}$, respectively, when $i = 1,3,5,7$. Furthermore, $A_{i\alpha\beta}$, $B_{i\alpha\beta}$, $\overline{A}_{i\alpha\beta}$, $\overline{B}_{i\alpha\beta}$ are set equal to zero when $i = 2,4,6,8$ for all α's and β's . This is understood, as stated in the preceding section, because the finite values of stresses and displacements must be maintained at $r = 0$. It must be mentioned here that, for the case of infinite length composites, the displacements and stresses must be finite as z approaches infinity; specifically, $A_{i\alpha}$ and $B_{i\alpha}$ when $i = 1,2,3,4$, $A_{\ell o}$ and $B_{\ell o}$ in Appendix I must be set to zero.

In Appendix I, all solutions for stresses and displacements are expressed in terms of Bessel functions or modified Bessel functions, depending on whether the μ's are real or imaginary. A constant k is defined as +1 whenever a Bessel function is used, or -1 whenever a modified Bessel function is adopted. The range and functions to be used will be discussed in the next section.

DOMAIN AND BOUNDARY CONDITIONS

There are three kinds of geometry for composite length that must be considered: finite, infinite, and semi-infinite length. The fiber array within the composite is assumed to be hexagonal. Each basic, representative element consists of a circular, cylindrical fiber surrounded by a shell matrix of circular section. The domains for these three cases are:

1. Finite Length Composite

(Fiber):
$$0 \le r \le a$$
$$0 \le z \le L$$
$$0 \le t \le \infty \tag{39}$$

(Matrix):
$$a \le r \le b$$
$$0 \le z \le L$$
$$0 \le t \le \infty \tag{40}$$

2. Infinite Length Composite

(Fiber):
$$0 \le r \le a$$
$$-\infty \le z \le \infty$$
$$0 \le t \le \infty \tag{41}$$

(Matrix):
$$a \le r \le b$$
$$-\infty \le z \le \infty$$
$$0 \le t \le \infty \tag{42}$$

3. Semi-Infinite Length Composite (Fiber): $0 \leq r \leq a$

$$0 \leq r \leq a, \quad 0 \leq z \leq \infty, \quad 0 \leq t \leq \infty \tag{43}$$

(Matrix):

$$a \leq r \leq b, \quad 0 \leq z \leq \infty, \quad 0 \leq t \leq \infty \tag{44}$$

For each element, the condition of perfect bonding between fiber and matrix and the compatibility conditions between basic representative elements must also be imposed. In other words, displacements and stresses σ_{ij} are continuous at the fiber-matrix interface, and all elements behave exactly alike. Therefore, the boundary conditions of an element at the lateral surfaces are:

1. At the interface,

$$\begin{aligned} u^{I}(a,z,t) &= u^{II}(a,z,t) \\ w^{I}(a,z,t) &= w^{II}(a,z,t) \\ \sigma_{11}^{I}(a,z,t) &= \sigma_{11}^{II}(a,z,t) \\ \sigma_{13}^{I}(a,z,t) &= \sigma_{13}^{II}(a,z,t) \end{aligned} \tag{45}$$

2. At the outer surface,

$$\begin{aligned} u^{II}(b,z,t) &= 0 \\ \sigma_{13}^{II}(b,z,t) &= 0 \end{aligned} \tag{46}$$

   The boundary conditions (46) are assumed such that all elements in an infinite region of composite vibrate simultaneously at the same phase and without longitudinal shear stresses between them.

3. All displacements and stresses should be finite as r approaches zero and/or t tends to infinite.

In addition to the boundary conditions stated above, more conditions are present for the different cases of vibration which will be considered here.

1. Case 1: Infinite and Finite Length, Free Vibration

   a. For the case of infinite length cylinders, all stresses and displacements should be finite as z approaches infinity.

b. For the case of finite length,

(1) At $z = 0$, fixed or free end,

$$\left\{\begin{aligned} w^{I,II}(r,0,t) &= 0 \\ \frac{\partial u^{I,II}}{\partial z}(r,0,t) &= 0 \end{aligned}\right. \quad \text{or} \quad \left\{\begin{aligned} \sigma_{33}^{I,II}(r,0,t) &= 0 \\ \sigma_{13}^{I,II}(r,0,t) &= 0 \end{aligned}\right. \tag{47}$$

The fixed end boundary conditions should actually be $w^{I,II} = 0$ and $u^{I,II} = 0$. The reason for using $\frac{\partial u^{I,II}}{\partial z}(r,0,t) = 0$ instead of $u^{I,II}(r,0,t) = 0$ is that this is a very good approximation if we want to have a consistent solution.

(2) At $z = L$, free end

$$\begin{aligned} \sigma_{33}^{I,II}(r,L,t) &= 0 \\ \sigma_{13}^{I,II}(r,L,t) &= 0 \end{aligned} \tag{48}$$

2. Case 2: Semi-Infinite Length With Free End

a. At $z = 0$,

$$\begin{aligned} \sigma_{33}^{I,II}(r,0,t) &= 0 \\ \sigma_{13}^{I,II}(r,0,t) &= 0 \end{aligned} \tag{49}$$

b. All stresses and displacements should tend to zero as z approaches infinity.

3. Case 3: Finite Length, Forced Vibration (One end, $z = 0$, is fixed, and the other end, $z = L$, is under axial piece-wise-constant or sinusoidal loading)

a. At $z = 0$,

$$\text{and} \quad \left.\begin{aligned} w^{I,II}(r,0,t) &= 0 \\ \frac{\partial u^{I,II}}{\partial z}(r,0,t) &= 0 \end{aligned}\right\} \tag{50}$$

b. At $z = L$,

$$2\pi \int_{0}^{a} \left[\sigma_{33}^{I}(r,L,t)\right] rdr +$$

$$2\pi \int_{a}^{b} \left[\sigma_{33}^{II}(r,L,t)\right] rdr = \begin{cases} P & \text{for } 0 < t < T/2 \\ -P & \text{for } -T/2 < t < 0 \end{cases} \quad \text{or} \tag{51}$$

$$= \begin{cases} P \sin(\omega_e t) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{52}$$

and

$$\sigma_{13}^{I,II}(r,L,t) = 0 \tag{53}$$

where T is period and ω_e is the external exciting frequency.

4. Case 4: Finite Length, Forced Vibration (One end, $z = 0$, is freely supported and the other end, $z = L$, is under axial piecewise-constant or sinusoidal loading)

a. At $z = 0$,

$$\begin{cases} \sigma_{33}^{I,II}(r,0,t) = 0 \\ \sigma_{13}^{I,II}(r,0,t) = 0 \end{cases} \tag{54}$$

b. At $z = L$,

$$2\pi \int_{0}^{a} \left[\sigma_{33}^{I}(r,L,t)\right] rdr +$$

$$2\pi \int_{a}^{b} \left[\sigma_{33}^{II}(r,L,t)\right] rdr = \begin{cases} P & \text{for } 0 < t < T/2 \\ -P & \text{for } -T/2 < t < 0 \end{cases} \quad \text{or} \tag{55}$$

$$= \begin{cases} P \sin(\omega_e t) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{56}$$

and

$$\sigma_{13}^{I,II}(r,L,t) = 0 \tag{57}$$

5. Case 5: Semi-Infinite Length, Forced Vibration (Axial Piece-wise-constant or sinusoidal loading applied at $z = 0$)

a. At $z = 0$,

$$2\pi \int_0^a \left[\sigma_{33}^{I}(r,0,t)\right] rdr +$$

$$2\pi \int_a^b \left[\sigma_{33}^{II}(r,0,t)\right] rdr = \begin{cases} P & \text{for } 0 < t < T/2 \\ -P & \text{for } -T/2 < t < 0 \end{cases} \quad \text{or} \tag{58}$$

$$= \begin{cases} P \sin(\omega_e t) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{59}$$

and

$$\sigma_{13}^{I,II}(r,0,t) = 0 \tag{60}$$

b. All stresses and displacements tend to zero as z approaches infinity.

CHARACTERISTIC EQUATION (FREQUENCY EQUATION) FOR THE CASES OF INFINITE AND FINITE LENGTH COMPOSITES

The domain, boundary conditions, and solutions of displacements and stresses are written in equations (39) through (42), (45) through (48) and the conditions thereunder, and in Appendix I.

For perfect bond in order to satisfy equation (45), the wave numbers along the axial direction and the circular frequencies of fiber and matrix must be identical; in other words,

$$\beta_1 = \beta_2 = \delta_1 = \delta_2 = \beta \tag{61}$$

$$\alpha_1 c_1^{II} = \alpha_2 c_2^{II} = \gamma_1 c_1^{I} = \gamma_2 c_2^{I} = \omega_\alpha \tag{62}$$

Then equations (29) and (31) become

$$\mu_{1\alpha\beta}^2 = \left(\frac{\omega_\alpha}{c_1^{II}}\right)^2 - \beta^2 = \left[\left(\frac{c_\alpha}{c_1^{II}}\right)^2 - 1\right]\beta^2 \tag{63}$$

$$\mu_{2\alpha\beta}^2 = \left(\frac{\omega_\alpha}{c_2^{II}}\right)^2 - \beta^2 = \left[\left(\frac{c_\alpha}{c_2^{II}}\right)^2 - 1\right]\beta^2 \tag{64}$$

Also,

$$\mu_{1\gamma\delta}^2 = \left(\frac{\omega_\alpha}{c_1^{I}}\right)^2 - \beta^2 = \left[\left(\frac{c_\alpha}{c_1^{I}}\right)^2 - 1\right]\beta^2 \tag{65}$$

$$\mu_{2\gamma\delta}^2 = \left(\frac{\omega_\alpha}{c_2^{I}}\right)^2 - \beta^2 = \left[\left(\frac{c_\alpha}{c_2^{I}}\right)^2 - 1\right]\beta^2 \tag{66}$$

where β is axial wave number, ω_α is circular frequency, and c_α is phase velocity.

Imposition of boundary conditions (45) and (46) onto equations (223) through (225) and (228) in Appendix I yields six simultaneous, homogeneous, algebraic equations. For a nontrivial solution of the amplitudes, the determinant of their coefficients is set equal to zero, resulting in the following characteristic equation:

$$\left|d_{ij}\right| = 0 \tag{67}$$

where $i,j = 1\ldots6$. This equation is written out in Appendix III to this report.

Equation (67) is a transcendental equation which relates circular frequency ω_γ to axial wave number β for composites of infinite and finite length.

As mentioned previously, all μ's may be either real or imaginary, depending on the circular frequency ω_α. Table I lists the range of circular frequency ω_α, the values of μ's, and the appropriate Bessel functions to be used in the expressions in which Bessel functions appear.

TABLE I

RANGE OF CIRCULAR FREQUENCIES AND APPROPRIATE BESSEL FUNCTIONS USED

Range of ω_α	Values of μ's	Appropriate Functions Used for Different Ranges of Circular Frequencies ω_α			
$\omega_\alpha > \beta c_1^{I}$	$\mu_{1\gamma\delta}, \mu_{2\gamma\delta}$ real	$J(\mu_{1\gamma\delta} r)$	0	$J(\mu_{2\gamma\delta} r)$	0
$\omega_\alpha > \beta c_1^{II}$	$\mu_{1\alpha\beta}, \mu_{2\alpha\beta}$ real	$J(\mu_{1\alpha\beta} r)$	$Y(\mu_{1\alpha\beta} r)$	$J(\mu_{2\alpha\beta} r)$	$Y(\mu_{2\alpha\beta} r)$
$\beta c_2^{I} < \omega_\alpha < \beta c_1^{I}$	$\mu_{1\gamma\delta}$ imaginary	$I(\bar{\mu}_{1\gamma\delta} r)$	0	--	--
	$\mu_{2\gamma\delta}$ real	--	--	$J(\mu_{2\gamma\delta} r)$	0
$\beta c_2^{II} < \omega_\alpha < \beta c_1^{II}$	$\mu_{1\alpha\beta}$ imaginary	$I(\bar{\mu}_{1\alpha\beta} r)$	$K(\bar{\mu}_{1\alpha\beta} r)$	--	--
	$\mu_{2\alpha\beta}$ real	--	--	$J(\mu_{2\alpha\beta} r)$	$Y(\mu_{2\alpha\beta} r)$
$\omega_\alpha < \beta c_2^{I}$	$\mu_{1\gamma\delta}, \mu_{2\gamma\delta}$ imaginary	$I(\bar{\mu}_{1\gamma\delta} r)$	0	$I(\bar{\mu}_{2\gamma\delta} r)$	0
$\omega_\alpha < \beta c_2^{II}$	$\mu_{1\alpha\beta}, \mu_{2\alpha\beta}$ imaginary	$I(\bar{\mu}_{1\alpha\beta} r)$	$K(\bar{\mu}_{1\alpha\beta} r)$	$I(\bar{\mu}_{2\alpha\beta} r)$	$K(\bar{\mu}_{2\alpha\beta} r)$

In equations (249) through (254), k is defined as before; in other words,

$$k = \begin{cases} +1 & \text{whenever it is associated with } J, Y \\ -1 & \text{whenever it is associated with } I, K \end{cases}$$

In principle, characteristic equation (67) should be valid for both composites of infinite length and of finite length, of a large aspect ratio. For a free-free cylinder, the stresses should vanish at both ends ($z = 0, L$); i.e.,

$$\sigma_{33}^{I,II}(r,0,t) = 0 \quad , \quad \sigma_{33}^{I,II}(r,L,t) = 0 \qquad (68)$$

With this in mind, after applying these boundary conditions (68) into equations (240) and (241) of Appendix I, we get

$$
\begin{aligned}
A_{5\alpha\beta} &= A_{7\alpha\beta} = A_{8\alpha\beta} = A_{9\alpha\beta} = B_{1\alpha\beta} = B_{3\alpha\beta} \\
&= B_{2\alpha\beta} = B_{4\alpha\beta} = C_{5\gamma\delta} = C_{7\gamma\delta} = D_{1\gamma\delta} = D_{3\gamma\delta} = 0 \\
A_{1\alpha} &= A_{2\alpha} = A_{3\alpha} = A_{4\alpha} = C_{1\gamma} = C_{3\gamma} \\
&= B_{1\alpha} = B_{2\alpha} = B_{3\alpha} = B_{4\alpha} = D_{1\gamma} = D_{3\gamma} = 0 \\
&\qquad B_{10} = B_{20} = 0
\end{aligned}
\tag{69}
$$

and

$$\beta(n) = \frac{n\pi}{L} \tag{70}$$

where $n = 1,2,3\ldots$. With the eigenvalues established through equation (70), we can find the exact values of circular frequency ω_α of a composite of finite length. It must be stated that the conditions of the vanishing shear stresses at both ends are not satisfied; in other words,

$$\sigma_{13}^{I,II}(r,0,t) \neq \sigma_{13}^{I,II}(r,L,t) \neq 0$$

This is not important, however, since shear stress σ_{13} is always small at both ends and self-equilibrating, the shear stress along the outside lateral boundary vanishes:

$$\sigma_{13}^{II}(b,z,t) = 0$$

CHARACTERISTIC EQUATION (FREQUENCY EQUATION) FOR THE CASE OF THE SEMI-INFINITE LENGTH COMPOSITE

In a similar manner, a system of six simultaneous, homogeneous algebraic equations is found by imposing boundary conditions (45) and (46) onto equations (242) through (244) and (247) in Appendix II. The vanishing of the determinant for the amplitude coefficients yields the frequency equation for the semi-infinite length composite, as follows:

$$\left|\bar{d}_{ij}\right| = 0 \tag{71}$$

where $i,j = 1\ldots6$. This equation is written out in Appendix IV.

It must be emphasized that equation (71) is very similar to equation (67); however, the physical meanings and mathematical results are different and should not be confused with each other. In equation (71), $\bar{\beta}$ is real and positive in all cases, but β in equation (67) has no such restriction and is the wave number in the axial direction for longitudinal vibration of the composite. Furthermore, μ's in the previous case may be real or imaginary, depending on the range of frequency; on the other hand, μ's in the semi-infinite rod are always real. In addition, equations (63) through (66) become

$$\mu_{1\alpha\beta}^2 = \left(\frac{\omega_\alpha}{c_1^{II}}\right)^2 + \bar{\beta}^2 \tag{72}$$

$$\mu_{2\alpha\beta}^2 = \left(\frac{\omega_\alpha}{c_2^{II}}\right)^2 + \bar{\beta}^2 \tag{73}$$

$$\mu_{1\gamma\delta}^2 = \left(\frac{\omega_\alpha}{c_1^{I}}\right)^2 + \bar{\beta}^2 \tag{74}$$

$$\mu_{2\gamma\delta}^2 = \left(\frac{\omega_\alpha}{c_2^{I}}\right)^2 + \bar{\beta}^2 \tag{75}$$

SOLUTIONS FOR FINITE LENGTH COMPOSITE WITH ONE END (z = 0) FIXED AND THE OTHER (z = L) SUBJECTED TO AXIAL PIECEWISE-CONSTANT OR SINUSOIDAL LOADING

Now let us solve a vibration problem of composite with one end fixed and the other end under piecewise-constant loading. No initial condition is specified, since only steady-state solution is obtained. In numerical calculation, period T as well as the magnitude of piecewise-constant loading P must be given.

The Fourier expansion of a piecewise-constant function P (equation 51) is

$$\frac{4P}{\pi} \sum_{n=1,2,3,\ldots}^{\infty} \frac{1}{2n-1} \sin \frac{2(2n-1)\pi t}{T} \tag{76}$$

Combining equations (50), (51),(53), (236), (237), (240), (241), and (76), we get $\left(\sigma_{13}{}^{I,II}(r,0,t) = 0\right)$

$$\frac{4P}{\pi}\left(\frac{1}{2n-1}\right) = -\left\{\sum_{\beta\geq o}^{\infty} A_{5\alpha\beta}\left[2\pi\int_{o}^{a}\left\{\left[\left(\lambda^{II} + 2G^{II}\right)\beta^{2} + k\lambda^{II}\,\bar{\mu}_{1\alpha\beta}^{2}\right]\cdot\right.\right.\right.$$

$$Z_{o}\left(\bar{\mu}_{1\alpha\beta}r\right) + M_{1\alpha\beta}\left[\left(\lambda^{II} + 2G^{II}\right)\beta^{2} + k\lambda^{II}\,\bar{\mu}_{1\alpha\beta}^{2}\right]\cdot$$

$$W_{o}\left(\bar{\mu}_{1\alpha\beta}r\right) - M_{2\alpha\beta}\left(2G^{II}\,\bar{\mu}_{2\alpha\beta}\,\beta\right)\cdot$$

$$Z_{o}\left(\bar{\mu}_{2\alpha\beta}r\right) - M_{3\alpha\beta}\left(2G^{II}\,\bar{\mu}_{2\alpha\beta}\,\beta\right) k\,W_{o}\left(\bar{\mu}_{2\alpha\beta}r\right)\Big\}\,rdr +$$

$$2\pi\int_{a}^{b}\left\{M_{4\alpha\beta}\left[\left(\lambda^{I} + 2G^{I}\right)\beta^{2} + k\lambda^{I}\,\bar{\mu}_{1\gamma\delta}^{2}\right]\cdot\right.$$

$$Z_{o}\left(\bar{\mu}_{1\gamma\delta}r\right) - M_{5\alpha\beta}\left(2G^{I}\,\bar{\mu}_{2\gamma\delta}\,\beta\right)\cdot$$

$$\left.\left.\left.Z_{o}\left(\bar{\mu}_{2\gamma\delta}r\right)\right\}rdr\right]\cos(\beta L)\right\} \tag{77}$$

$$\omega_{n} = \frac{2(2n-1)\pi}{T} \tag{78}$$

where $n = 1,2,3,\ldots$, and the rest of the coefficients in the expressions for stresses and displacements are zeroes (reference equations 262, 263, 267, and 269 in Appendix V). Here in equation (77), M's and μ's are defined by equations (266) and (270) through (276).

In order to obtain the coefficients $A_{5\alpha\beta}$, we must employ the so-called quasi-orthogonality property [24] of a function across multiple media. The eigenfunctions in equation (77) are not orthogonal over the total interval in the radial direction, because the conditions of the Sturm-Liouville problem are violated, in that the physical constants of the

governing differential equations of a composite are different for each constituent. Therefore, it is not possible to represent a function as the expansions of such nonorthogonal eigensets in the ordinary sense, such as Fourier-Bessel or Dini-Bessel expansion.

In general, the Fourier coefficients F_p of a function $Q(r)$ for a multiple M-layer composite can be determined by

$$F_p = \left\{ \sum_{m=1}^{M} \chi_m^2 \int_{r_{mi}}^{r_{mo}} r\, Q(r)\, R\left(\mu_p r\right) dr \right\} \div \left\{ \sum_{m=1}^{M} \chi_m^2 \int_{r_{mi}}^{r_{mo}} r\, R^2\left(\mu_p r\right) dr \right\} \qquad (79)$$

where χ_m is defined as

$$\sum_{m=1}^{M} \chi_m^2 \sum_{q \neq p}^{\infty} \int_{r_{mi}}^{r_{mo}} r \left[R\left(\mu_p r\right)\right] \left[R\left(\mu_q r\right)\right] dr = 0 \qquad (80)$$

where M is the number of layers and the m^{th} region is $r_{mi} \leq r \leq r_{mo}$. Equation (80) is the condition of the quasi-orthogonality and R = eigenfunction corresponding to homogeneous boundary conditions of the type considered in the problem.

For the present problem, the coefficients $A_{5\alpha\beta}$ of equation (77) may be represented in the following form:

$$A_{5\alpha\beta} \cos(\beta L) = -\frac{4P}{\pi} \left\{ \left\{ \chi_1^2 \left(\frac{1}{2n-1}\right) 2\pi \int_0^a \left[\int_0^a \left\{ M_{4\alpha\beta} \left[\left(\lambda^I + 2G^I\right) \beta^2 + \right.\right.\right.\right.\right.$$

$$\left. k\lambda^I \bar{\mu}_{1\gamma\delta}^2 \right] Z_0\left(\bar{\mu}_{1\gamma\delta} r\right) - M_{5\alpha\beta}\left(2G^I \bar{\mu}_{2\gamma\delta}\, \beta\right) \cdot$$

$$\left. Z_0\left(\bar{\mu}_{2\gamma\delta} r\right)\right\} rdr \Bigg] rdr \Bigg] + \left[\chi_2^2 \left(\frac{1}{2n-1}\right) \cdot \right.$$

$$2\pi \int_a^b \left[\int_a^b \left\{ \left[\left(\lambda^{II} + 2G^{II}\right) \beta^2 + k\lambda^{II} \bar{\mu}_{1\alpha\beta}^2 \right] Z_0\left(\bar{\mu}_{1\alpha\beta} r\right) + \right.\right.$$

$$+ M_{1\alpha\beta}\left[\left(\lambda^{II} + 2G^{II}\right)\beta^2 + k\lambda^{II}\bar{\mu}_{1\alpha\beta}^2\right] W_o\left(\bar{\mu}_{1\alpha\beta}r\right) -$$

$$M_{2\alpha\beta}\left(2G^{II}\bar{\mu}_{2\alpha\beta}\beta\right) Z_o\left(\bar{\mu}_{2\alpha\beta}r\right) -$$

$$\left.\left.\left.\left.M_{3\alpha\beta}\left(2G^{II}\bar{\mu}_{2\alpha\beta}\beta\right) k W_o\left(\bar{\mu}_{2\alpha\beta}r\right)\right\} r'dr'\right] rdr\right)\right\} \div$$

$$\left(\chi_1^2\int_0^a\left[2\pi\int_0^a\left\{M_{4\alpha\beta}\left[\left(\lambda^I + 2G^I\right)\beta^2 + k\lambda^I\bar{\mu}_{1\gamma\delta}^2\right] Z_o\left(\bar{\mu}_{1\gamma\delta}r\right)r'dr' -\right.\right.\right.$$

$$\left.\left.M_{5\alpha\beta}\left(2G^I\bar{\mu}_{2\gamma\delta}\beta\right) Z_o\left(\bar{\mu}_{2\gamma\delta}r\right)r'dr'\right\}\right]^a rdr +$$

$$\chi_2^2\int_a^b\left[2\pi\int_a^b\left\{\left[\left(\lambda^{II} + 2G^{II}\right)\beta^2 + k\lambda^{II}\bar{\mu}_{1\alpha\beta}^2\right] Z_o\left(\bar{\mu}_{1\alpha\beta}r\right) r'dr' +\right.\right.$$

$$M_{1\alpha\beta}\left[\left(\lambda^{II} + 2G^{II}\right)\beta^2 + k\lambda^{II}\bar{\mu}_{1\alpha\beta}^2\right] W_o\left(\bar{\mu}_{1\alpha\beta}r\right) r'dr' -$$

$$M_{2\alpha\beta}\left(2G^{II}\bar{\mu}_{2\alpha\beta}\beta\right) Z_o\left(\bar{\mu}_{2\alpha\beta}r\right) r'dr' -$$

$$\left.\left.\left.M_{3\alpha\beta}\left(2G^{II}\bar{\mu}_{2\alpha\beta}\beta\right) W_o\left(\bar{\mu}_{2\alpha\beta}r\right) r'dr'\right]_o^a\right\} rdr\right) \quad (81)$$

where χ_1 and χ_2 are defined in equations 280 in Appendix V.

With $A_{5\alpha\beta}$ found by equation (81) and with the eigenvalues obtained from equations (248) through (254), we can get $A_{5\alpha\beta}$; $B_{1\alpha\beta}$, $C_{1\gamma\delta}$, $D_{1\gamma\delta}$ from equations (270) through (276) and then obtain displacements and stresses of the composite from equations (236) through (241). This completes the solutions for composites of finite length with one end (z = 0) fixed and the other end (z = L) subjected to axial piecewise-constant loading. Detailed procedures are written out in Appendix V.

If the composite is under sinusoidal loading $P \sin(\omega_e t)$ at $z = L$, the problem is much easier to solve. From equations (52) and (240), we have the following:

$$P = -A_5 \left\{ 2\pi \int_a^b \left\{ \left[\left(\lambda^{II} + 2G^{II} \right) \beta^2 + k\lambda^{II} \bar{\mu}_{1\alpha\beta}^2 \right] Z_0 \left(\bar{\mu}_{1\alpha\beta} r \right) \cdot \right. \right.$$

$$M_1 \left[\left(\lambda^{II} + 2G^{II} \right) \beta^2 + k\lambda^{II} \bar{\mu}_{1\alpha\beta}^2 \right] W_0 \left(\bar{\mu}_{1\alpha\beta} r \right) -$$

$$M_2 \left(2G^{II} \bar{\mu}_2 \beta \right) Z_0 \left(\bar{\mu}_{2\alpha\beta} r \right) -$$

$$M_3 \left(2G^{II} \bar{\mu}_2 \beta \right) k\, W_0 \left(\bar{\mu}_{2\alpha\beta} r \right) \Big\} rdr +$$

$$2\pi \int_0^a \left\{ M_4 \left[\left(\lambda^I + 2G^I \right) \beta^2 + k\lambda^I \bar{\mu}_{1\gamma\delta}^2 \right] Z_0 \left(\bar{\mu}_{1\gamma\delta} r \right) - \right.$$

$$\left. \left. M_5 \left(2G^I \bar{\mu}_{2\gamma\delta} \beta \right) Z_0 \left(\bar{\mu}_{2\gamma\delta} r \right) \right\} rdr \right\} \qquad (82)$$

$$\omega_\alpha = \omega_e \qquad (83)$$

and the other coefficients vanish. In equation (82), M's and μ's are defined in equations (266) and (270) through (276). It should be mentioned that all of the subscripts associated with this case should be dropped, since summation is not performed in this problem. Also, it should be noted that, for a composite under longitudinal loading of simple harmonic force, the Fourier series expansion and the quasi-orthogonality technique of the function are not used.

A_5 in equation (82) can be determined by the integration of both sides. Therefore, other coefficients of solutions under Appendix I can be found from equation (270).

SOLUTIONS FOR FINITE LENGTH COMPOSITES WITH ONE END (z = 0) FREELY SUPPORTED AND THE OTHER (z = L) SUBJECTED TO AXIAL PIECEWISE-CONSTANT OR SINUSOIDAL LOADING

From the following equation,

$$P = 2\pi\left[\int_0^a \left(\sigma_{33}^{I}\right)_{z=L} rdr + \int_a^b \left(\sigma_{33}^{II}\right)_{z=L} rdr\right] \tag{84}$$

and proceeding in the same manner delineated in the previous section, we can obtain the expression for the coefficient $A_{1\alpha\beta}$ in the case of a composite placed under piecewise loading.

$$A_{1\alpha\beta}\sin(\beta L) = -\frac{4P}{\pi}\left\{\left[\!\left[\chi_3^2\left(\frac{1}{2n-1}\right)\cdot\right.\right.\right.$$

$$2\pi\int_0^a\left[\int_0^a\left\{M_{9\alpha\beta}\left[\left(\lambda^I + 2G^I\right)\beta^2 + k\lambda^I\,\bar{\mu}_{2\gamma\delta}^2\right] Z_o\left(\bar{\mu}_{1\gamma\delta}\right) +\right.\right.$$

$$\left.\left.\left. M_{10\alpha\beta}\left(2G^I\,\bar{\mu}_{2\gamma\delta}\,\beta\right) Z_o\left(\bar{\mu}_{2\gamma\delta} r\right)\right\} r'dr'\right] rdr\right]\!\!\right] + \left[\!\left[\chi_4^2\left(\frac{1}{2n-1}\right)\cdot\right.\right.$$

$$2\pi\int_0^b\left[\int_0^b\left\{\left[\!\left[\left(\lambda^{II} + 2G^{II}\right)\beta^2 + k\lambda^{II}\,\bar{\mu}_{1\alpha\beta}^2\right] Z_o\left(\bar{\mu}_{1\alpha\beta} r\right) +\right.\right.\right.$$

$$M_{6\alpha\beta}\left[\left(\lambda^{II} + 2G^{II}\right)\beta^2 + k\lambda^{II}\,\bar{\mu}_{1\alpha\beta}^2\right] W_o\left(\bar{\mu}_{1\alpha\beta} r\right) +$$

$$M_{7\alpha\beta}\left(2G^{II}\,\bar{\mu}_{2\alpha\beta}\,\beta\right) Z_o\left(\bar{\mu}_{2\alpha\beta} r\right) +$$

$$\left.\left.\left.\left. M_{8\alpha\beta}\left(2G^{II}\,\bar{\mu}_{2\alpha\beta}\,\beta\right) k\, W_o\left(\bar{\mu}_{2\alpha\beta} r\right)\right\} r'dr'\right] rdr\right]\!\!\right]\right\} \div$$

$$\left[\!\left[\chi_3^2\int_0^a\left[2\pi\int_0^a\left\{M_{9\alpha\beta}\left[\left(\lambda^I + 2G^I\right)\beta^2 + k\lambda^I\,\bar{\mu}_{1\gamma\delta}^2\right] Z_o\left(\bar{\mu}_{1\gamma\delta} r\right) r'dr' +\right.\right.\right.$$

$$+ M_{10\alpha\beta}\left(2G^{I}\bar{\mu}_{2\gamma\delta}\,\beta\right) Z_o\left(\bar{\mu}_{2\gamma\delta} r\right) r'dr' \Big\} \Big]^2 rdr +$$

$$\chi_4^2 \int_a^b \Big[2\pi \int_a^b \Big\{ \Big[\left(\lambda^{II} + 2G^{II}\right)\beta^2 + k\lambda^{II}\,\bar{\mu}_{1\alpha\beta}^2 \Big] Z_o\left(\bar{\mu}_{1\alpha\beta} r\right) r'dr' +$$

$$M_{6\alpha\beta}\Big[\left(\lambda^{II} + 2G^{II}\right)\beta^2 + k\lambda^{II}\,\bar{\mu}_{1\alpha\beta}^2\Big] W_o\left(\bar{\mu}_{1\alpha\beta} r\right) r'dr' +$$

$$M_{7\alpha\beta}\left(2G^{II}\bar{\mu}_{2\alpha\beta}\,\beta\right) Z_o\left(\bar{\mu}_{2\alpha\beta} r\right)' r'dr' +$$

$$M_{8\alpha\beta}\left(2G^{II}\bar{\mu}_{2\alpha\beta}\,\beta\right) W_o\left(\bar{\mu}_{2\alpha\beta} r\right) r'dr' \Big\} \Big]^2 rdr \Big\} \quad (85)$$

where χ_3 and χ_4 are defined by equation (299).

With $A_{1\alpha\beta}$ found by equation (85) and with the eigenvalues obtained from equations (248) through (254), we can obtain $A_{2\alpha\beta}$, $B_{5\alpha\beta}$, $B_{6\alpha\beta}$, $C_{1\gamma\delta}$, $D_{5\gamma\delta}$ from equations (289) through (295), and then obtain the stresses and displacements of the composite from equations (236) through (241). This completes the solutions for composites of finite length with one end (z = 0) freely supported and the other (z = L) subjected to axial piecewise-constant loading. Detailed procedures are written out in Appendix VI.

For the case of a composite cylinder element subjected to sinusoidal loading $P \sin(\omega_e t)$, we have

$$\omega_\alpha = \omega_e \quad (86)$$

$$P = -A_1 \Big(2\pi \int_a^b \Big\{ \Big[\left(\lambda^{II} + 2G^{II}\right)\beta^2 + k\lambda^{II}\,\bar{u}_{1\alpha\beta}^2 \quad Z_o\left(\bar{u}_{1\alpha\beta} r\right)\Big] +$$

$$M_6\Big[\left(\lambda^{II} + 2G^{II}\right)\beta^2 + k\lambda^{II}\,\bar{u}_{1\alpha\beta}^2\Big] W_o\left(\bar{u}_{1\alpha\beta} r\right) +$$

$$M_7\left(2G^{II}\bar{u}_{2\alpha\beta}\beta\right) Z_o\left(\bar{u}_{2\alpha\beta} r\right) + M_8\left(2G^{II}\bar{u}_{2\alpha\beta}\beta\right) k\, W_o\left(\bar{u}_{2\alpha\beta} r\right)\Big\} rdr \; -$$

$$- 2\pi \int_0^a \Bigg\{ M_9 \left[\left(\lambda^I + 2G^I\right)\beta^2 + k\lambda^I \bar{\mu}^2_{1\gamma\delta} \right] Z_0\left(\bar{\mu}_{1\gamma\delta} r\right) +$$

$$M_{10}\left(2G^I \bar{\mu}_{2\gamma\delta}\beta\right) Z_0\left(\bar{\mu}_{2\gamma\delta} r\right) \Bigg\} rdr \Bigg) \qquad (87)$$

where M's are defined in equation (289), with the subscripts dropped. Once the other coefficients are obtained, we can then calculate stresses and displacements from equations (236) through (241) without the need to perform a double summation.

SOLUTIONS FOR SEMI-INFINITE LENGTH COMPOSITES WITH THE END z = 0 UNDER AXIAL PIECEWISE-CONSTANT OR SINUSOIDAL LOADING

Combining boundary conditions (58), (45), and (46) with equations (242) through (244) and (247), together with the use of generalized Fourier series techniques, the coefficients $\bar{A}_{5\alpha\beta}$ can be found as follows:

$$\bar{A}_{5\alpha\beta} = -\frac{4P}{\pi} \Bigg\| \Bigg(\bar{X}_1^2 \left(\frac{1}{2n-1}\right) 2\pi \int_0^a \left[\int_0^a \Bigg\{ \bar{M}_{4\alpha\beta} \left[\left(\lambda^I + 2G^I\right)\bar{\beta}^2 - \lambda^I \mu^2_{1\gamma\delta} r \right] \cdot$$

$$J_0\left(\mu_{1\gamma\delta} r\right) - \bar{M}_{5\alpha\beta}\left(2G^I \mu_{2\gamma\delta}\bar{\beta}\right) J_0\left(\mu_{2\gamma\delta} r\right) \Bigg\} r'dr' \right] rdr \Bigg) +$$

$$\Bigg(\bar{X}_2^2 \left(\frac{1}{2n-1}\right) 2\pi \int_a^b \left[\int_a^b \Bigg\{ \left[\left(\lambda^{II} + 2G^{II}\right)\bar{\beta}^2 - \lambda^{II} \mu^2_{1\alpha\beta} \right] \cdot$$

$$J_0\left(\mu_{1\alpha\beta} r\right) + \bar{M}_{1\alpha\beta}\left[\left(\lambda^{II} + 2G^{II}\right)\bar{\beta}^2 - \lambda^{II} \mu^2_{1\alpha\beta} \right] \cdot$$

$$Y_0\left(\mu_{1\alpha\beta} r\right) - \bar{M}_{2\alpha\beta}\left(2G^{II} \mu_{2\alpha\beta}\bar{\beta}\right) \cdot$$

$$J_0\left(\mu_{2\alpha\beta} r\right) - \bar{M}_{3\alpha\beta}\left(2G^{II} \mu_{2\alpha\beta}\bar{\beta}\right) Y_0\left(\mu_{2\alpha\beta} r\right) \Bigg\} r'dr' \right] rdr \Bigg) \Bigg\| \div$$

$$\div \left(\bar{\chi}_1^2 \int_o^a \left[2\pi \int_o^a r' \left\{ \bar{M}_{4\alpha\beta} \left[\left(\lambda^I + 2G^I \right) \bar{\beta}^2 - \lambda^I \, u_{1\gamma\delta}^2 \right] \cdot \right. \right. \right.$$

$$J_o\left(u_{1\gamma\delta} r\right) dr - \bar{M}_{5\alpha\beta} \left(2G^I \, u_{2\gamma\delta} \bar{\beta}\right) J_o\left(u_{2\gamma\delta} r\right) \Big\} dr' \Big]^2 rdr +$$

$$\bar{\chi}_2^2 \int_a^b \left[2\pi \int_a^b r' \left\{ \left[\left(\lambda^{II} + 2G^{II} \right) \bar{\beta}^2 - \lambda^{II} \, u_{1\alpha\beta}^2 \right] \cdot \right. \right.$$

$$J_o\left(u_{1\alpha\beta} r\right) \; dr' + \bar{M}_{1\alpha\beta} \left[\left(\lambda^{II} + 2G^{II} \right) \bar{\beta}^2 - \lambda^{II} \, u_{1\alpha\beta} \right] \cdot$$

$$Y_o\left(u_{1\alpha\beta} r\right) \; dr' - \bar{M}_{2\alpha\beta} \left(2G^{II} \, u_{3\alpha\beta} \bar{\beta}\right) \cdot$$

$$J_o\left(u_{2\alpha\beta} r\right) \; dr' - \bar{M}_{3\alpha\beta} \left(2G^{II} \, u_{2\alpha\beta} \bar{\beta}\right) \; Y_o\left(u_{2\alpha\beta} r\right) \Big\} dr' \Big]^2 dr \Bigg) \qquad (88)$$

where $\bar{\chi}_1$ and $\bar{\chi}_2$ are defined by equation (314).

With $\bar{A}_{5\alpha\beta}$ found by equation (88) and the eigenvalues obtained from equations (255) through (260), we can get $\bar{A}_{6\alpha\beta}$, $\bar{B}_{5\alpha\beta}$, $\bar{B}_{6\alpha\beta}$, $\bar{C}_{5\gamma\delta}$, and $D_{5\gamma\delta}$, from equations (304) through (310) and then obtain the solutions of the stresses and displacements of the composite from equations (242) through (247). This completes the solutions for composites of semi-infinite length with the end $z = 0$ subjected to axial piecewise-constant loading. Detailed procedures and solutions are written out in Appendix VII.

For a composite of semi-infinite length with the end $z = 0$ subjected to sinusoidal loading, we have, by applying boundary conditions (56) in equation (246),

$$P = -A_5 \left(2\pi \int_a^b \left\{ \left[\left(\lambda^{II} + 2G^{II} \right) \bar{\beta}^2 - \lambda^{II} \, u_{1\alpha\beta}^2 \right] J_o\left(u_{1\alpha\beta} r\right) + \right. \right.$$

$$\bar{M}_1 \left[\left(\lambda^{II} + 2G^{II} \right) \bar{\beta}^2 - \lambda^{II} \, u_{1\alpha\beta}^2 \right] \; Y_o\left(u_{1\alpha\beta} r\right) -$$

$$\bar{M}_2 \left(2G^{II} \, u_{2\alpha\beta} \bar{\beta}\right) J_o\left(u_{2\alpha\beta} r\right) -$$

$$\bar{M}_3 \left(2G^{II} \, u_{2\alpha\beta} \bar{\beta}\right) Y_o\left(u_{2\alpha\beta} r\right) \Big\} rdr +$$

$$+ 2\pi \int_0^a \left\{ \bar{M}_4 \left[\left(\lambda^I + 2G^I\right)\bar{\beta}^2 - \lambda^I \mu_{1\gamma\delta}^2 \right] J_o\left(\mu_{1\gamma\delta} r\right) - \right.$$

$$\left. \bar{M}_5 \left(2G^I \mu_{2\gamma\delta} \bar{\beta}\right) J_o\left(\mu_{2\gamma\delta} r\right) \right\} rdr \Bigg] \quad (89)$$

where $\bar{M}$'s and μ's are determined by equations (301) and (304) through (310). As was done in the previous section, A_5 can be found by integrating both ends from $r = 0$ to $r = b$. The other coefficients, $\bar{A}_5$, $\bar{B}_5$, B_5, C_5, and $\bar{D}_5$, are then obtained from equations (304). The solutions of the stresses and displacements are therefore obtainable from Appendix II.

CONCLUSIONS AND DISCUSSION

Frequency equations which relate circular frequencies and axial wave numbers have been established for the cases of infinite and finite, and semi-infinite length composites (reference Appendixes III and IV). For longitudinal forced vibration, the solutions of stresses and displacements have been obtained for composites of the following types:

1. Finite length cylinder with one end ($z = 0$) fixed and the other end ($z = L$) under axial piecewise-constant or sinusoidal loading.

2. Finite length cylinder with one end ($z = 0$) freely supported and the other ($z = L$) under axial piecewise-constant or sinusoidal loading.

3. Semi-infinite length composite with the end $z = 0$ under axial piecewise constant or sinusoidal loading.

These solutions are given in Appendixes V, VI, and VII, together with Appendixes I and II.

In obtaining the steady state solution of forced vibration, the generalized Fourier series technique was used instead of transforms or Green functions. The method devised for this program is much simpler than these techniques. The transform or Green function methods would be required for the study of the behavior of transient phenomena, however.

PART II

ANALYSIS OF THE VIBRATIONS OF A COMPOSITE MATERIAL IN STEADY AND TRANSIENT STATE USING AN APPROXIMATE THEORY

For real composite materials, the ratio of fiber length to fiber diameter is large (more than 1000). This suggests that it could be possible to analyze the composite vibrations for waves traveling in the fiber direction by using an additional hypothesis of deformation, as it is assumed for bars under longitudinal vibrations. This additional hypothesis of deformation is the so-called Bernoulli hypothesis; namely, that the plane cross sections remain plane while the wave is passing through.

To assure that results obtained from a composite materials theory based on the Bernoulli hypothesis will be accurate, the criterion will be analogous to that established for the longitudinal vibration of bars; in other words, that the accuracy decreases when the ratio of fiber diameter to wave length increases. The approximate theory will be applicable even for the study of high-frequency vibrations, since the fiber diameter is quite small. Part III of this report will compare numerical results obtained from the exact Navier's equations for typical composite materials (Part I of this report) and those obtained by using the approximate theory. This comparison will define the field of application of the approximate theory. It can be said in advance, however, that the results obtained from this theory are sufficiently accurate to be applicable to most of the technical problems encompassed by this program.

The boundary conditions are simplified, as in Part I, by assuming symmetry of revolution. However, the constants that appear in the fundamental differential equation are also computed by considering the more exact boundary conditions corresponding to the hexagonal arrangement of the fiber into the matrix. Part III gives numerical results for the comparison of both types of boundary conditions.

The basic representative element used in the development of the approximate theory is identical to that used in Part I (Figure 1) as are the hypothesis for the material characteristics (e.g., perfect elasticity, isotropy).

By using the approximate theory, we can find the velocity of the propagation of elastic waves, and the solutions for the free-free and free-fixed end cases, in both steady and transient states of vibration. These results are extended to encompass the semi-infinite composite. In the transient state, both impact and sudden loads are given consideration.

The finite Fourier transforms and the Laplace transforms are employed in the integration of the differential equation. These transforms represent the space and time variables, respectively. The corresponding solutions are given in the form of Fourier series.

DERIVATION OF THE FUNDAMENTAL DIFFERENTIAL EQUATION

The so-called Bernoulli hypothesis is assumed. A slice of a composite with a thickness Δz is considered (see Figure 2).

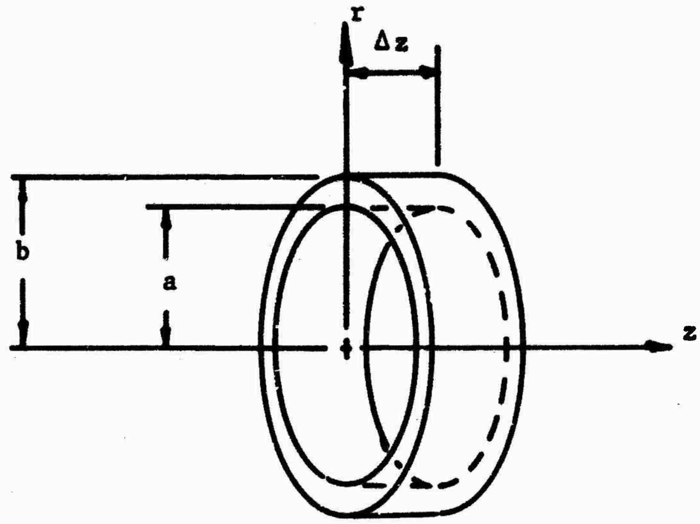

Figure 2. Basic Element

When the wave front passes through this slice, all the plane cross sections remain plane, on the basis of the Bernoulli hypothesis. There is then a plane strain at any plane cross section $z = z_0$, or, in another form,

$$(\epsilon_{33})_{z=z_0} = f(t) \tag{90}$$

At any cross section, the following boundary conditions must also be satisfied:

$$\left(\sigma_{11}^{I}\right)_{r=a} = \left(\sigma_{11}^{II}\right)_{r=a}$$

$$\left(u^{I}\right)_{r=a} = \left(u^{II}\right)_{r=a}$$

$$\left(u^{II}\right)_{r=b} = 0 \tag{91}$$

which are equivalent to the boundary conditions given by the expressions in equations (45) and (46).

From the displacements $u = C_1 r + \frac{C_2}{r}$, we obtain the strains

$$\epsilon_{11} = C_1 - \frac{C_2}{r^2} \quad , \quad \epsilon_{22} = C_1 + \frac{C_2}{r^2} \tag{92}$$

By setting

$$C_1^{I} = K_1^{I}\epsilon_{33} \quad , \quad C_1^{II} = K_1^{II}\epsilon_{33} \quad , \quad C_2^{II} = K_2^{II}\epsilon_{33} \tag{93}$$

the total displacement can be obtained

$$u^{I} = \left(-\nu^{I} + K_1^{I}\right)\epsilon_{33} r \tag{94}$$

and, for the matrix,

$$u^{II} = \left(-\nu^{II} + K_1^{II}\right)\epsilon_{33} r + K_2^{II}\frac{\epsilon_{33}}{r} \tag{95}$$

The total specific strain energy per unit of length is

$$\overline{W}^{I} = \left\{\frac{E^{I}}{2}\frac{E^{I}}{\left(1+\nu^{I}\right)\left(1-2\nu^{I}\right)}\left[\left(K_1^{I}\right)^2 + \left(1-2\nu^{I}\right)\frac{\left(K_1^{I}\right)^2}{r^4}\right]\right\}\epsilon_{33}^2 \tag{96}$$

$$\overline{W}^{II} = \left\{\frac{E^{II}}{2} + \frac{E^{II}}{\left(1+\nu^{II}\right)\left(1-2\nu^{II}\right)}\left[\left(K_1^{II}\right)^2 + \left(1-2\nu^{II}\right)\frac{\left(K_2^{II}\right)^2}{r^4}\right]\right\}\epsilon_{33}^2 \tag{97}$$

To establish the differential equation of motion from Hamilton's variational principle, it is necessary to know the kinetic and potential energies of the system.

The kinetic energy per unit length is

$$T = \frac{1}{2}\int_V \rho V^2 dV = \frac{1}{2}\left\{\int_0^a \rho^I \left[\left(\frac{\partial u^I}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] 2\pi r\, dr + \right.$$

$$\left. \int_a^b \rho^{II} \left[\left(\frac{\partial u^{II}}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] 2\pi r\, dr \right\}$$

$$= \Omega_1 \left(\frac{\partial^2 w}{\partial z\, \partial t}\right)^2 + \Omega_2 \left(\frac{\partial w}{\partial t}\right)^2 \tag{98}$$

where the equations (94) and (95) have been used and where, after performing the integration and extensive analysis, the constants Ω_1 and Ω_2 were defined as follows:

$$\Omega_1 = \rho^I \pi \left(-\nu^I + K_1^I\right)^2 \frac{a^4}{4} + \rho^{II} \pi \left[\left(-\nu^{II} + K_1^{II}\right)^2 \frac{b^4 - a^4}{4} + \right.$$

$$\left. \left(-\nu^{II} + K_1^{II}\right) K_2^{II}\left(b^2 - a^2\right) + \left(K_2^{II}\right)^2 \ln \frac{b}{a}\right]$$

$$\Omega_2 = \rho^I \pi \frac{a^2}{2} + \rho^{II} \pi \frac{b^2 - a^2}{2} \tag{99}$$

The potential energy is

$$W = \int_V \mathcal{W} dV = \left[\frac{E^I}{2} \int_o^a 2\pi r dr + \frac{E^{II}}{2} \int_a^b 2\pi r dr \right] \left(\frac{\partial w}{\partial z}\right)^2 +$$

$$\frac{E^I}{\left(1+\nu^I\right)\left(1-2\nu^I\right)} \left(\frac{\partial w}{\partial z}\right)^2 \int_o^a \left(K_1^I\right)^2 2\pi r dr +$$

$$\frac{E^{II}}{\left(1+\nu^{II}\right)\left(1-2\nu^{II}\right)} \left(\frac{\partial w}{\partial z}\right)^2 \int_a^b \left[\left(K_1^{II}\right)^2 + \left(1-2\nu^{II}\right) \frac{\left(K_2^{II}\right)^2}{r^4} \right] 2\pi r dr$$

$$= \Omega_3 \left(\frac{\partial w}{\partial z}\right)^2 \tag{100}$$

where

$$\Omega_3 = \pi \left\{ \frac{E^I}{2} a^2 + \frac{E^{II}}{2} \left(b^2 - a^2\right) + \frac{E^I}{\left(1+\nu^I\right)\left(1-2\nu^I\right)} \left(K_1^I\right)^2 a^2 + \right.$$

$$\left. \frac{E^{II}}{\left(1+\nu^{II}\right)\left(1-2\nu^{II}\right)} \left(K_1^{II}\right)^2 \left(b^2 - a^2\right) - \frac{E^{II}}{1+\nu^{II}} \left(K_2^{II}\right)^2 \left(\frac{1}{b^2} - \frac{1}{a^2}\right) \right\} \tag{101}$$

Having established the energies of the system, it is now possible to apply Hamilton's variational principle so that we can obtain the Euler-Lagrange equation. This will be the differential equation which describes the motion of the system. The expression of the Hamilton principle is

$$\delta \iint (T-W) dzdt = \delta \iint F dzdt = 0 \tag{102}$$

where

$$F = \Omega_1 \left(\frac{\partial^2 w}{\partial z\, \partial t}\right)^2 + \Omega_2 \left(\frac{\partial w}{\partial t}\right)^2 - \Omega_3 \left(\frac{\partial w}{\partial z}\right)^2 \tag{103}$$

Equation (103) is obtained from equations (98) and (100). The variation of the integral (102) gives the Euler-Lagrange equation directly:

$$-\frac{\partial}{\partial t} \frac{\partial F}{\partial\left(\frac{\partial w}{\partial t}\right)} + \frac{\partial^2}{\partial t\, \partial z} \frac{\partial F}{\partial\left(\frac{\partial^2 w}{\partial t\, \partial z}\right)} - \frac{\partial}{\partial z} \frac{\partial F}{\partial\left(\frac{\partial w}{\partial z}\right)} = 0 \tag{104}$$

Introducing equation (103) into (104), the fundamental differential equation is obtained:

$$\Omega_1 \frac{\partial^4 w}{\partial z^2 \partial t^2} - \Omega_2 \frac{\partial^2 w}{\partial t^2} + \Omega_3 \frac{\partial^2 w}{\partial z^2} = 0 \tag{105}$$

The constants Ω_1 , Ω_2 , and Ω_3 are given in equations (99) and (101).

By performing the variation of equation (102), other differential equations appear in addition to (105). These are the natural boundary conditions.

The initial boundary condition is

$$\frac{\partial F}{\partial\left(\frac{\partial w}{\partial t}\right)} - \frac{\partial}{\partial z} \frac{\partial F}{\partial\left(\frac{\partial^2 w}{\partial z\, \partial t}\right)} = 0 \tag{106}$$

or, expressing F by equation (103),

$$\Omega_2 \frac{\partial w}{\partial t} - \Omega_1 \frac{\partial^3 w}{\partial z^2 \partial t} = 0 \tag{107}$$

where t is constant.

The condition at the bar ends is

$$\frac{\partial F}{\partial\left(\frac{\partial w}{\partial z}\right)} - \frac{\partial}{\partial t} \frac{\partial F}{\partial\left(\frac{\partial^2 w}{\partial z\, \partial t}\right)} = 0 \tag{108}$$

or introducing F from equation (101),

$$\eta_1 \frac{\partial^3 w}{\partial z\, \partial t} + \eta_0 \frac{\partial w}{\partial z} = 0$$

The fundamental differential equation (103) will be solved for different conditions. First, the steady-state waves will be considered.

If it is specified the total force P acting on one end of the composite

$$P = 2\pi \int_0^a \sigma_{33}^{I}\, rdr + 2\pi \int_0^a \sigma_{33}^{II}\, rdr ,$$

by using the Hooke's law it is possible to write

$$P = K\, \epsilon_{33} ,$$

with

$$K = \pi \left\{ \left[\frac{2\nu^{I} K_1^{I}}{\left(1+\nu^{I}\right)\left(1-2\nu^{I}\right)} + 1 \right] E^{I} a^2 + \left[\frac{2\nu^{II} K_1^{II}}{\left(1+\nu^{II}\right)\left(1-2\nu^{II}\right)} + 1 \right] E^{II} (b^2 - a^2) \right\} \qquad (109)$$

STEADY STATE OF VIBRATIONS

To find the phase velocity, we assume a sinusoidal displacement

$$w = A \sin \frac{2\pi}{\lambda} (z - ct) \tag{110}$$

where A is the amplitude, λ is the wavelength, and c is the wave propagation velocity. Introducing equation (110) into (105) yields the following equation for the wave propagation velocity in a composite:

$$c = \sqrt{\frac{\Omega_3}{\Omega_2 + \frac{4\pi^2}{\lambda^2}}} \tag{111}$$

Therefore, in a composite material, the wave velocity depends not only on the material of the components and the geometry involved, but also on the wavelength λ .

Taking into account the frequency

$$f = \frac{\omega}{2\pi} = \frac{c}{\lambda}$$

we express equation (111) in the following form:

$$c = \sqrt{\frac{\Omega_3 - \omega^2 \Omega_1}{\Omega_2}} = \sqrt{\frac{\Omega_3 - 4\pi^2 f^2 \Omega_1}{\Omega_2}} \tag{112}$$

If the lateral inertia is neglected, then Ω_1 is zero, and c does not depend on the frequency f .

Part III of this report presents the numerical calculations needed to establish the velocity of propagation for different types of composites, using expressions (111) and (112).

The two steady state cases are treated in the following analysis. One is the fixed-free composite with an exciting harmonic load at the free end, as depicted in Figure 3.

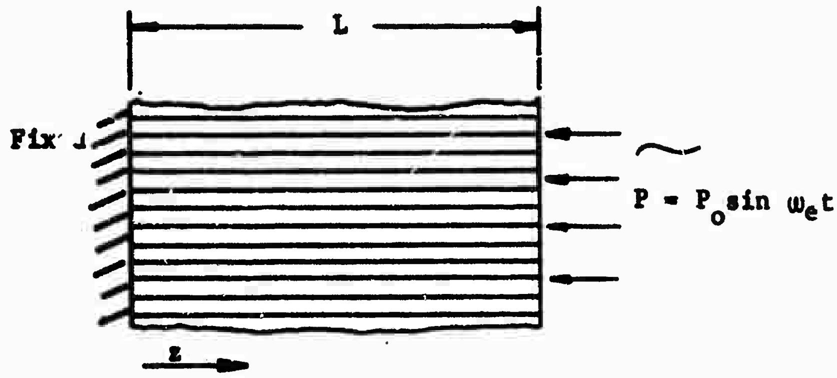

Figure 3. Fixed-Free Composite Under Periodic Load

The second is a free-free composite with an exciting harmonic load, as shown in Figure 4.

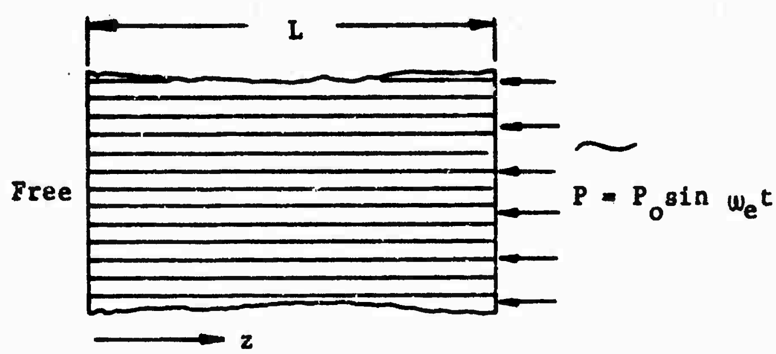

Figure 4. Free-Free Composite Under Periodic Force

Assuming the solution of differential equation (105) is

$$w(z,t) = \varphi(z) \sin \omega_e t \tag{113}$$

then equation (105) becomes

$$\frac{d^2\varphi}{dz^2} + \frac{\Omega_2 \omega^2}{\Omega_3 - \omega^2 \Omega_1} \varphi = 0 \tag{114}$$

The solution of this ordinary differential equation is

$$\varphi = B_1 \sin \sqrt{\frac{\Omega_2 \omega^2}{\Omega_3 - \omega^2 \Omega_1}}\, z + B_2 \cos \sqrt{\frac{\Omega_2 \omega^2}{\Omega_3 - \omega^2 \Omega_1}}\, z \qquad (115)$$

for $\Omega_3 - \omega^2 \Omega_1 > 0$.

The following boundary conditions will be used to determine B_1 and B_2 .

$$\text{Fixed-Free} \quad \begin{cases} \varphi(0) = 0 \\ \left.\dfrac{d\varphi}{dz}\right|_{z=L} = \dfrac{P_o}{K} \end{cases} \qquad (116)$$

$$\text{Free-Free} \quad \begin{cases} \left.\dfrac{d\varphi}{dz}\right|_{z=o} = 0 \\ \left.\dfrac{d\varphi}{dz}\right|_{z=L} = \dfrac{P_o}{K} \end{cases} \qquad (117)$$

where K is a constant. From equations (115) through (117), we obtain, for the fixed-free case,

$$\varphi = \frac{P_o}{K} \sqrt{\frac{\Omega_3 - \omega^2 \Omega_1}{\Omega_2 \omega^2}} \; \frac{\sin \sqrt{\dfrac{\Omega_2 \omega_e^2}{\Omega_3 - \omega_e^2 \Omega_1}}\, z}{\cos \sqrt{\dfrac{\Omega_2 \omega_e^2}{\Omega_3 - \omega_e^2 \Omega_1}}\, L} \qquad (118)$$

and, for the free-free case,

$$\varphi = \frac{P_o}{K} \sqrt{\frac{\Omega_3 - \omega^2 \Omega_1}{\Omega_2 \omega^2}} \; \frac{\cos \sqrt{\dfrac{\Omega_3 \omega_e^2}{\Omega_3 - \omega_e^2 \Omega_1}}\, z}{\sin \sqrt{\dfrac{\Omega_2 \omega_e^2}{\Omega_3 - \omega_e^2 \Omega_1}}\, L} \qquad (119)$$

To find the natural frequencies, the denominator of equations (118) and (119) must be zero. Therefore, in the fixed-free case, the natural frequencies are

$$\omega_n = \frac{(2n+1)\pi}{L} \sqrt{\frac{\Omega_3}{4\Omega_2 + \frac{(2n+1)^2}{L^2} \pi^2 \Omega_1}} \tag{120}$$

and, in the free-free case,

$$\omega_n = \frac{n\pi}{L} \sqrt{\frac{\Omega_3}{\Omega_2 + \frac{n^2 \pi^2}{L^2} \Omega_1}} \tag{121}$$

In the event that the exciting force is periodic but not harmonic, then expressions similar to (118) and (119) can be applied for each component of the Fourier series development of the periodic force.

Once Φ is determined by using equations (118) or (119), the displacement w is obtained from equation (113). By differentiation with respect to z, we find ϵ_{33}, and from this, we can compute the stresses with the expression (92). The stress σ_{33} for the fiber and the matrix, respectively, results:

$$\begin{aligned} \sigma_{33}^{I} &= E^{I} \epsilon_{33} + \nu^{I} \left(\sigma_{11}^{I} + \sigma_{22}^{I} \right) \\ \sigma_{33}^{II} &= E^{II} \epsilon_{33} + \nu^{II} \left(\sigma_{11}^{II} + \sigma_{22}^{II} \right) \end{aligned} \tag{122}$$

The stress distribution for some composites is shown in Part III of this report.

TRANSIENT STATE OF VIBRATIONS

For the transient state of vibrations, the differential equation (105) must be solved by considering not only boundary conditions but also initial conditions.

To begin, the finite cosine Fourier transform to the fundamental equation (105) is applied.

$$\Omega_1 \int_0^L \frac{\partial^4 w}{\partial z^2 \partial t^2} \cos\frac{n\pi z}{L}\, dz - \Omega_2 \int_0^L \frac{\partial^2 w}{\partial t^2} \cos\frac{n\pi z}{L}\, dz +$$

$$\Omega_3 \int_0^L \frac{\partial^2 w}{\partial z^2} \cos\frac{n\pi z}{L}\, dz = 0 \qquad (123)$$

If

$$\bar{w}_n(t) = \int_0^L w(z,t) \cos\frac{n\pi z}{L}\, dz \qquad (124)$$

is the finite cosine Fourier transform, then equacion (123) becomes

$$\left(\Omega_1 \frac{n^2\pi^2}{L^2} + \Omega_2\right)\frac{d^2\bar{w}_n}{dt^2} + \Omega_3 \frac{n^2\pi^2}{L^2}\bar{w}_n = \Omega_1(-1)^n \frac{\partial^3 w}{\partial z \partial t^2}(L,t) +$$

$$\Omega_3(-1)^n \frac{\partial w}{\partial z}(L,t) -$$

$$\Omega_1 \frac{\partial^3 w}{\partial z\, \partial t^2}(o,t) -$$

$$\Omega_3 \frac{\partial w}{\partial z}(o,t) \qquad (125)$$

This is an ordinary differential equation in $\bar{w}_n(t)$. To solve this equation, the Laplace transform will be applied.

As the first case, the problem of a composite of finite length and of free-free character (Figure 4) placed under a sudden applied load on z = L will be considered. Figures 5 through 7 are indicative of the variation of the load, and show its first and second derivatives with respect to time.

The boundary conditions are

$$\frac{\partial w}{\partial z}(o,t) = 0 \quad \text{for} \quad t \geq 0 \tag{126}$$

$$\frac{\partial w}{\partial z}(L,t) = \begin{cases} 0 & \text{for} \quad t \leq 0 \\ \frac{P}{K} & \text{for} \quad t > 0 \end{cases} \tag{127}$$

The Laplace transform of the function $\overline{w}_n(t)$ is

$$f_n(p) = \int_0^\infty \overline{w}_n(t)\, e^{-pt} dt \tag{128}$$

The transformation of the second derivative of $\overline{w}_n(t)$ is

$$\int_0^\infty \frac{d^2\overline{w}_n(t)}{dt^2} e^{-pt} dt = p^2 f_n(p) - p\overline{w}_n(o^-) - \frac{d\overline{w}_n(o^-)}{dt} \tag{129}$$

If the composite is at rest before load is applied,

$$\overline{w}_n(o^-) = \frac{d\overline{w}_n(o^-)}{dt} = 0 \tag{130}$$

With equations (126) through (130), equation (125) becomes

$$\left[\Omega_1 \frac{n^2\pi^2}{L^2} + \Omega_2 p^2\right] f_n(p) + \Omega_3 \frac{n^2\pi^2}{L^2} f_n(p) = (-1)^n \frac{P}{K}\left\{\Omega_3 \frac{1}{p} + \Omega_1 p\right\} \tag{131}$$

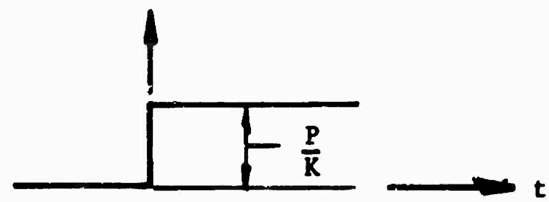

Figure 5. Variation of Load

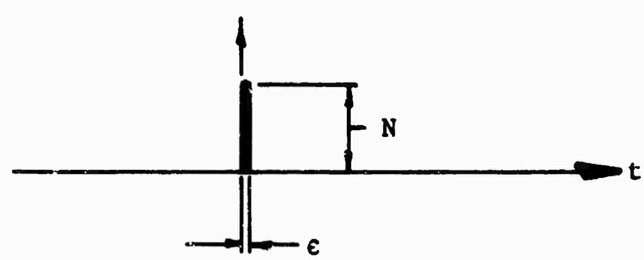

Figure 6. First Derivative
($N \times \varepsilon = 1$)

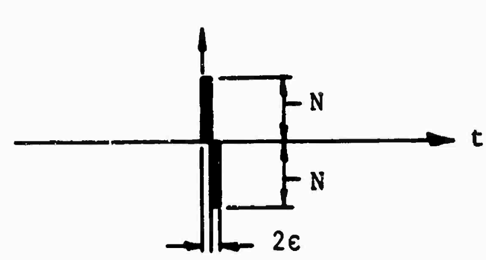

Figure 7. Second Derivative
($N \times \varepsilon^2 = 1$)

Solving equation (131) for $f_n(p)$, we obtain

$$f_n(p) = \frac{P}{K} \times \frac{(-1)^n \Omega_3}{\Omega_1 \dfrac{n^2\pi^2}{L^2} + \Omega_2} \times \frac{1}{p\left(p^2 + \dfrac{\Omega_3}{\Omega_1 + \dfrac{L^2}{n^2\pi^2}\Omega_2}\right)} +$$

$$\frac{P}{K} \times \frac{(-1)^n \Omega_3}{\Omega_1 \dfrac{n^2\pi^2}{L^2} + \Omega_2} \times \frac{p}{p^2 + \dfrac{\Omega_3}{\Omega_1 + \dfrac{L^2}{n^2\pi^2}\Omega_2}} \tag{132}$$

Applying the inverse Laplace transform to equation (132) we obtain, after some manipulation,

$$\bar{w}_n(t) = \frac{P}{K}(-1)^n \frac{L^2}{\pi^2 n^2}\left\{1 - \left(1 - \frac{n^2\pi^2}{L^2}\frac{\Omega_1}{\Omega_2 + \Omega_1 \dfrac{n^2\pi^2}{L^2}}\right)\cos \frac{n\pi\sqrt{\Omega_3}\,t}{\Omega_1\pi^2 n^2 + L^2\Omega_2}\right\} \tag{133}$$

For $n \to 0$, the last expression becomes indeterminate. Applying the L'Hospital rule, we have

$$\bar{w}_o(t) = \frac{P}{K}\frac{\Omega_3}{\Omega_2}\left\{\frac{\Omega_1}{\Omega_3} + \frac{t^2}{2}\right\} \tag{134}$$

The inverse of the finite cosine Fourier transform is given by:

$$w(z,t) = \frac{1}{L}\bar{w}_o(t) + \frac{2}{L}\sum_{n=1}^{\infty} \bar{w}_n(t) \cos\frac{n\pi z}{L} \tag{135}$$

Introducing equations (133) and (134) into (135) finally leads to the following equation for the displacement:

$$w(z,t) = \frac{P}{K\ell}\frac{\Omega_3}{\Omega_2}\left\{\frac{\Omega_1}{\Omega_3} + \frac{t^2}{2}\right\} +$$

$$\frac{2P\ell}{K\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi z}{L}\left\{1 - \frac{1}{1 + n^2\frac{\pi^2}{L^2}\frac{\Omega_1}{\Omega_2}}\cos\frac{n\pi\sqrt{\frac{\Omega_3}{\Omega_2}}\,t}{L\sqrt{1+\frac{\pi^2 n^2}{L^2}\frac{\Omega_1}{\Omega_2}}}\right\} \tag{136}$$

In this equation, we see a constant term which, in general, is a very small displacement of the whole system in the z direction. Another term is proportional to time squared, corresponding to the action of the constant force over the system considered as a rigid body. The other terms of equation (136) represent an infinite number of wave displacements.

In the next portion, an impact load will be considered. In this case, the boundary condition (126) is also valid. Instead of boundary condition (127), however, we now have the following (illustrated in Figure 8).

$$\frac{\partial w}{\partial z}(L,t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{P}{K} & \text{for } 0 \le t \le \epsilon \\ =0 & \text{for } t > \epsilon \end{cases} \tag{137}$$

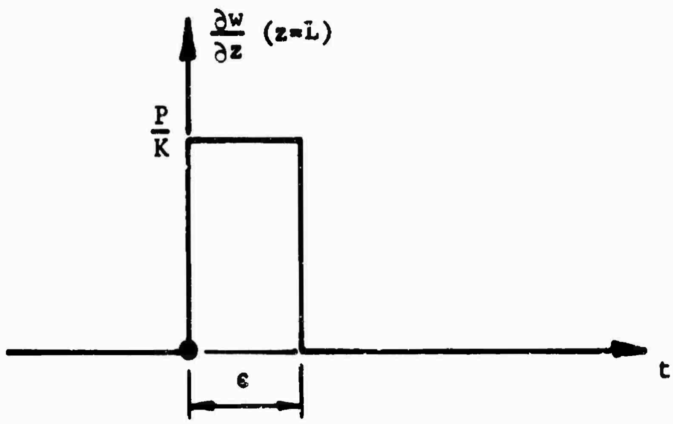

Figure 8. Boundary Condition for Impact

The Laplace transform of equation (137) is

$$\mathcal{L}\, \frac{\partial w}{\partial z}(L,t) = \frac{P}{K}\,\frac{1 - e^{-\epsilon p}}{p} \tag{138}$$

Consequently,

$$\mathcal{L}\left(\frac{\partial^3 w}{\partial z\,\partial t^2}\right)(L,t) = \frac{P}{K}\, p \left(1 - e^{-\epsilon p}\right) \tag{139}$$

The impulse can be constructed by adding the two-step function a and b , shown in Figure 9, displaced to each other by the time ϵ .

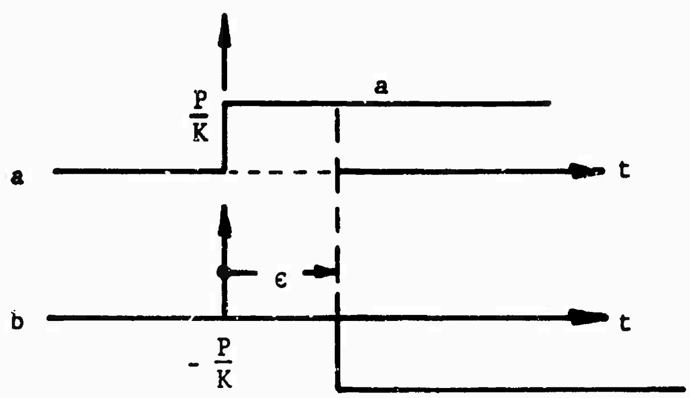

Figure 9. Forming the Step Function

From this, the Laplace transform is obtained by simple multiplication of equation (132) by the factor $1 - e^{-\epsilon p}$. Thus, the Laplace transform is

$$f_n(p) = (-1)^n \frac{P}{K}\,\frac{\Omega_1}{\Omega_1 \frac{n^2\pi^2}{L^2} + \Omega_2}\left\{\frac{1 - e^{-\epsilon p}}{p\left(p^2 + \frac{n^2\pi^2}{L^2}\,\frac{\Omega_3}{\Omega_2 + \Omega_1 \frac{n^2\pi^2}{L^2}}\right)}\right\} +$$

$$(-1)^n \frac{P}{K}\,\frac{\Omega_1}{\Omega_1 \frac{n^2\pi^2}{L^2} + \Omega_2}\left\{\frac{p(1 - e^{-\epsilon p})}{p^2 + \frac{n^2\pi^2}{L^2}\,\frac{\Omega_3}{\Omega_2 + \Omega_1 \frac{n^2\pi^2}{L^2}}}\right\} \tag{140}$$

Then the function $w(z,t)$ is equal to equation (136) for $0 < t \leq \epsilon$.

For $\epsilon < t \leq \infty$, $w(z,t)$ is obtained by subtracting the same function with the variable $(t - \epsilon)$ in place of t , from equation (136):

$$w(z,t) = \frac{P}{2K}\frac{\Omega_3 \epsilon}{I\Omega_2}(2t - \epsilon) +$$

$$\frac{2P\ell}{Kn^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\frac{\cos\frac{n\pi z}{L}}{1+n^2\frac{\pi^2}{L^2}\frac{\Omega_1}{\Omega_2}}\left\{\cos\frac{n\pi\sqrt{\frac{\Omega_3}{\Omega_2}}(t-\epsilon)}{L\sqrt{1+\frac{\pi^2 n^2\Omega}{L^2\Omega^2}}} - \cos\frac{n\pi\sqrt{\frac{\Omega_3}{\Omega_2}}\,t}{L\sqrt{1+\frac{\pi^2 n^2\Omega_1}{L^2\Omega_2}}}\right\} \tag{141}$$

If the time of impact is small, ϵ is moving toward zero, and equation (141) becomes

$$w(z,t) = \frac{I}{K}\frac{\Omega_3}{I\Omega_2}\,t +$$

$$\frac{2I}{K\pi}\sqrt{\frac{\Omega_3}{\Omega_2}}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{\cos\frac{n\pi z}{L}\sin\left(\frac{n\pi\sqrt{\frac{\Omega_3}{\Omega_2}}\,t}{L\sqrt{1+\frac{\pi^2 n^2}{L^2}\frac{\Omega_1}{\Omega_2}}}\right)}{\left(1+n^2\frac{\pi^2}{L^2}\frac{\Omega_1}{\Omega_2}\right)^{\frac{3}{2}}} \tag{142}$$

where $I = P\epsilon$.

To compute the impact momentum, we must assume that the impacting mass is an ideal, rigid body. Then, the impact velocity is equal to the velocity V of the mass

$$\frac{\partial w}{\partial t}(L,o) = V \tag{143}$$

Introducing equation (142) into equation (143) and solving for I , we get

$$I = \frac{VLK\Omega_2}{\Omega_3} \frac{1}{1 + 2 \sum_{n=1}^{\infty} \left(\frac{1}{1 + n^2 \frac{\pi^2}{L^2} \frac{\Omega_1}{\Omega_2}} \right)^2}$$

or expressing the summation in terms of hyperbolic functions,

$$I = \frac{2V \frac{K}{L} \frac{\Omega_1}{\Omega_3} \sinh^2 \left(L \sqrt{\frac{\Omega_2}{\Omega_1}} \right)}{1 + \frac{1}{2L} \sqrt{\frac{\Omega_1}{\Omega_2}} \sinh \left(2L \sqrt{\frac{\Omega_2}{\Omega_1}} \right)} \tag{144}$$

where the displacement w due to the impact momentum I in a free-free composite is given by (142). If the impacting mass is not a perfectly rigid body, the right side of equation (144) must be multiplied by a factor $0 \leq \alpha \leq 1$.

We will now consider the fixed-free end composite (Figure 3). We first consider an applied impulsive load (Figure 8). Then the boundary conditions are

$$w(o,t) = 0$$

$$\frac{\partial w}{\partial z}(L,t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ \frac{P}{K} & \text{for} \quad 0 < t \leq \varepsilon \\ 0 & \text{for} \quad t > \varepsilon \end{cases} \tag{145}$$

Taking into account that, from equation (145),

$$\mathcal{L}\left[\frac{\partial w}{\partial z}(L,t)\right] = \frac{P}{K}\frac{1}{p}(1 - e^{-\varepsilon p})$$

$$\mathcal{L}\left[\frac{\partial^3 w}{\partial z \partial t^2}(L,t)\right] = \frac{P}{k} p (1 - e^{-\varepsilon p}) \tag{146}$$

and calling

$$\left.\begin{aligned} \frac{\partial w}{\partial z}(0,t) &= F(t) \\ \mathcal{L}\left[\frac{\partial w}{\partial z}(0,t)\right] &= \phi(p) \\ \mathcal{L}\left[\frac{\partial^3 w}{\partial z \partial t^2}(0,t)\right] &= p^2\phi(p) \end{aligned}\right\} \tag{147}$$

the Laplace transform of $\overline{w}_n(t)$ results in

$$f_n(p) = \frac{\left[(-1)^n \frac{P}{K}(1 - e^{-\epsilon p}) - \phi(p)\right]\left[\Omega_3 + \Omega_1 p^2\right]}{p\left[\Omega_3 \frac{n^2\pi^2}{L^2} + \left(\Omega_1 \frac{n^2\pi^2}{L^2} + \Omega_2\right) p^2\right]}\ , \tag{148}$$

where

$$f_n(p) = \int_0^\infty \overline{w}_n(t)\, e^{-pt}\, dt \tag{149}$$

Remembering that

$$\overline{w}_n(t) = \mathcal{L}^{-1}[f_n(p)] \tag{150}$$

$$w(z,t) = \frac{1}{L}\overline{w}_0(t) + \frac{2}{L}\sum_{n=1}^{\infty}\overline{w}_n(t)\cos\frac{n\pi z}{L} \tag{151}$$

and in accordance with the first equation (145), we have for $z = 0$,

$$0 = \frac{1}{L}\bar{w}_o(t) + \frac{2}{L}\sum_{n=1}^{\infty}\bar{w}_n(t) \tag{152}$$

In other words,

$$\mathcal{L}^{-1}\left[f_o(p)\right] = -2\sum_{n=1}^{\infty}\mathcal{L}^{-1}\left[f_n(p)\right] = -2\mathcal{L}^{-1}\left[\sum_{n=1}^{\infty} f_n(p)\right] \tag{153}$$

Taking the Laplace transform of this equation, we obtain

$$f_o(p) = -2\sum_{n=1}^{\infty} f_n(p) \tag{154}$$

Substituting equation (148) into equation (154), we hold

$$\frac{\frac{P}{K}(1 - e^{-\epsilon p}) - \phi(p)}{\Omega_2 p^2} = -2\sum_{n=1}^{\infty}\frac{(-1)^n \frac{P}{K}(1 - e^{-\epsilon p}) - \phi(p)}{\Omega_3 \frac{n^2\pi^2}{L^2} + \left(\Omega_1 \frac{n^2\pi^2}{L^2} + \Omega_2\right)p^2} \tag{155}$$

and solving for $\phi(p)$:

$$\phi(p) = \frac{P}{K}(1 - e^{-\epsilon p})\frac{1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{1 + \frac{\pi^2}{L^2}\frac{\Omega_3 + p^2\Omega_1}{\Omega_2 p^2}n^2}}{1 + 2\sum_{n=1}^{\infty}\frac{1}{1 + \frac{\pi^2}{L^2}\frac{\Omega_3 + p^2\Omega_1}{\Omega_2 p^2}n^2}} \tag{156}$$

Considering the identities

$$\sum_{n=1}^{\infty} \frac{1}{1 + a^2 n^2} = \frac{1}{2}\left(\frac{\Pi}{a} \operatorname{cth} \frac{\Pi}{a} - 1\right) \tag{157}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1 + a^2 n^2} = \frac{1}{2}\left(\frac{\Pi}{a} \frac{1}{\operatorname{sh} \frac{\Pi}{a}} - 1\right) \tag{158}$$

where a is a constant, then equation **(156)** becomes

$$\phi(p) = \frac{P}{K}\left(1 - e^{-\epsilon p}\right) \frac{1}{\operatorname{ch}\left(\frac{L\sqrt{\Omega_2}\, p}{\sqrt{\Omega_3 + p^2 \Omega_1}}\right)} \tag{159}$$

Now we may calculate the inverse Laplace transform of

$$\varphi(p) = \frac{\phi(p)}{\frac{P}{K}\left(1 - e^{-\epsilon p}\right)} = \frac{1}{\operatorname{ch}\left(\frac{L\sqrt{\Omega_2}\, p}{\sqrt{\Omega_3 + p^2 \Omega_1}}\right)} = \frac{1}{\operatorname{ch}\left(\frac{bp}{\sqrt{p^2 + a^2}}\right)} \tag{160}$$

with

$$\sqrt{\frac{\Omega_3}{\Omega_1}} = a \quad , \quad L\sqrt{\frac{\Omega_2}{\Omega_1}} = b \tag{161}$$

Using the Bromwhich integral, we have

$$f(t) = \mathcal{L}^{-1}\left[\varphi(p)\right] = \lim_{y \to \infty} \frac{1}{2\pi i} \int_{\alpha - iy}^{\alpha + iy} \frac{e^{pt}}{\operatorname{ch}\left(\frac{bp}{\sqrt{p^2 + a^2}}\right)} dp \tag{162}$$

It is possible to put $\alpha = 0$ because the real part of p is less than zero ($Re[p] \leq 0$) for $t > 0$.

To solve the complex integral of equation (162), we will apply the residues theorem. The denominator of the integrand has poles (Figure 10) at

$$p_n = \pm ia \frac{1}{\sqrt{1 + \frac{4b^2}{\pi^2 (2n + 1)^{1/2}}}} \tag{163}$$

where $n = 0, \pm 1, \pm 2, \ldots$.

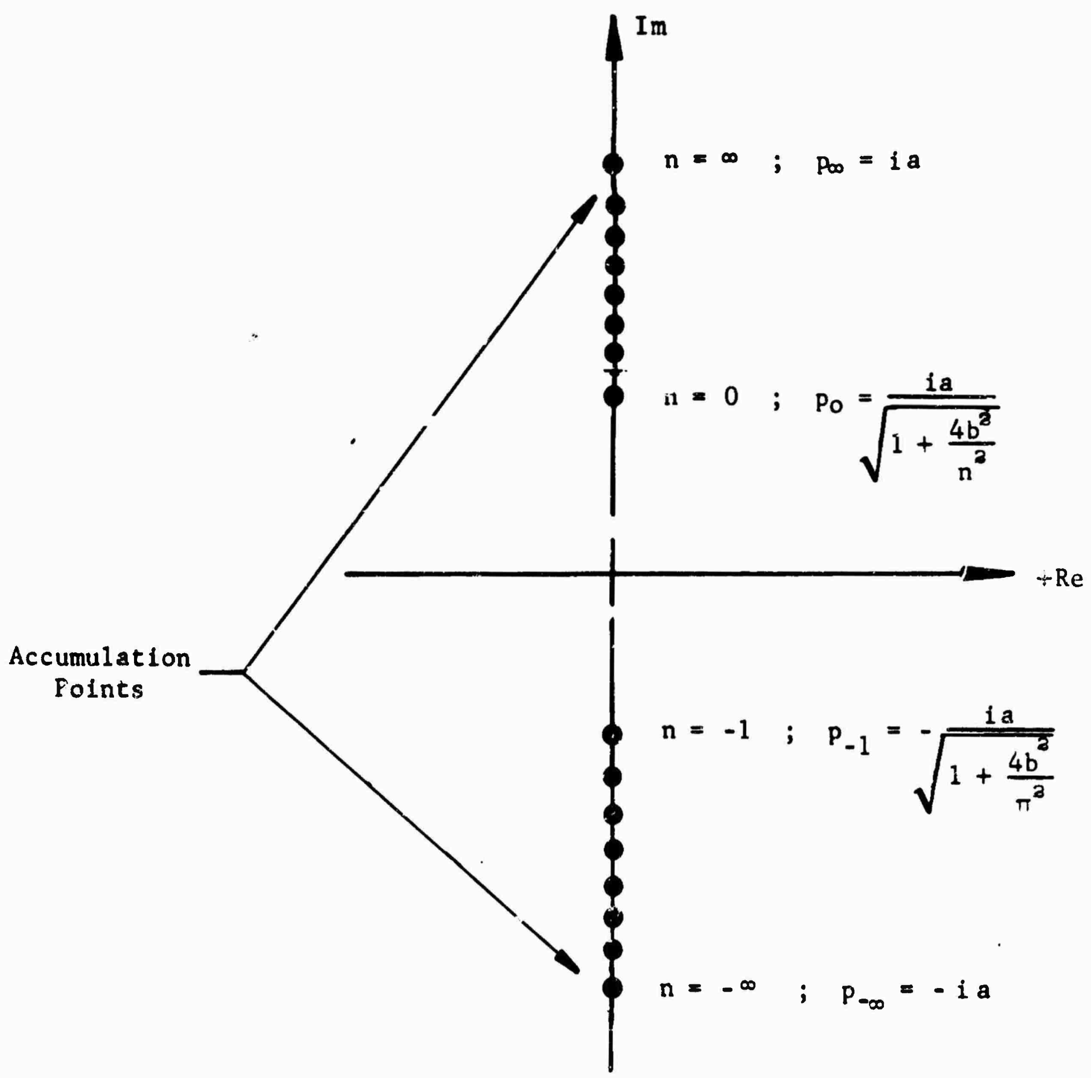

Figure 10. Poles for Laplace Inversion

The residue for an arbitrary n is given by:

$$\lim_{p \to p_n} \frac{e^{pt}(p - p_n)}{\operatorname{ch}\left(\frac{bp}{\sqrt{p^2 + a^2}}\right)} = (-1)^n \frac{a}{ib} \frac{e^{p_n t}}{\left\{1 + \left[\frac{(2n + 1)\pi}{2b}\right]^2\right\}^{3/2}} \tag{164}$$

To find this result, the L'Hospital rule has been applied. Using equation (164), we obtain the sum of all the residues:

$$\frac{2a}{b} \sum_{n=0}^{\infty} (-1)^n \frac{\sin \dfrac{at}{\left[1 + \dfrac{4b^2}{(2n + 1)^2 \pi^2}\right]^{1/2}}}{\left\{1 + \left[\frac{(2n + 1)\pi}{2b}\right]^2\right\}^{3/2}} \tag{165}$$

Replacing the integral of equation (162) with the value given in equation (165), and remembering equations (161), we obtain

$$f(t) = \frac{2}{L}\sqrt{\frac{\Omega_3}{\Omega_2}} \sum_{\nu=0}^{\infty} (-1)^\nu \frac{\sin\left\{\dfrac{t\sqrt{\dfrac{\Omega_3}{\Omega_1}}}{\left[1 + \dfrac{4L^2}{\pi^2}\dfrac{\dfrac{\Omega_2}{\Omega_1}}{(2\nu + 1)^2}\right]^{1/2}}\right\}}{\left\{1 + \dfrac{\Omega_1}{\Omega_2}\left[\dfrac{(2\nu + 1)\pi}{2L}\right]^2\right\}^{3/2}} \tag{166}$$

To perform the inversion of equation (148), we write this equation in the following form:

$$f_n(p) = \frac{(-1)^n \frac{P}{K}(1 - e^{-\epsilon p})\left[\Omega_3 + \Omega_1 p^2\right]}{p\left[\Omega_3 \frac{n^2\pi^2}{L^2} + \left(\Omega_1 \frac{n^2\pi^2}{L^2} + \Omega_2\right)p^2\right]} -$$

$$\phi(p) \frac{\Omega_3 + \Omega_1 p^2}{p\left[\Omega_3 \frac{n^2\pi^2}{L^2} + \left(\Omega_1 \frac{n^2\pi^2}{L^2} + \Omega_2\right)p^2\right]} \tag{167}$$

and we find the inverse of the second term by putting

$$\mathcal{L}^{-1}\left\{\frac{\phi(p)}{\frac{P}{K}\,(1-e^{-\epsilon p})}\cdot\frac{\frac{P}{K}\,(1-e^{-\epsilon p})\left(\Omega_3+\Omega_1 p^2\right)}{p\left[\Omega_3\,\frac{n^2\pi^2}{L^2}+\left(\Omega_1\,\frac{n^2\pi^2}{L^2}+\Omega_2\right)p^2\right]}\right\}$$

$$=\int_0^t f(\tau)\,\bar{w}_n(t-\tau)\,d\tau \qquad (168)$$

because $f(\tau)$, given by equation (166), is the inverse transformation of $\varphi(p)$, as equation (162) illustrates. $w_n\,(t-\tau)$ is the inverse transformation of the second factor; it is given by equation (142) , with $t-\tau$ instead of t:

$$\bar{w}_n(t-\tau) = -\frac{P\epsilon L}{K\pi}\,\frac{\Omega_3}{\Omega_2}\,\frac{(-1)^n}{n}\,\frac{\sin\left\{\frac{n\pi}{L}\cdot\frac{\sqrt{\frac{\Omega_3}{\Omega_2}}\,(t-\tau)}{\sqrt{1+\frac{\pi^2 n^2}{L^2}\,\frac{\Omega_1}{\Omega_2}}}\right\}}{\left(1+\frac{n^2\pi^2}{L^2}\,\frac{\Omega_1}{\Omega_2}\right)^{3/2}} \qquad (169)$$

$$\bar{w}_0(t-\tau) = -\frac{P\epsilon}{K}\,\frac{\Omega_3}{\Omega_2}\,(t-\tau) \qquad (170)$$

where $n=1,2,\ldots$. Thus, substituting equations (169) and (170) into equation (168), we can find:

For $n \neq 0$:

$$\mathcal{L}^{-1}\left[\frac{\phi(p)\;(\Omega_3+\Omega_1 p^2)}{p\left[\Omega_3\,\frac{n^2\pi^2}{L^2}+\left(\Omega_1\,\frac{n^2\pi^2}{L^2}+\Omega_2\right)p^2\right]}\right] = -\frac{2P\epsilon}{K\pi}\,\frac{\Omega_2}{\Omega_3}\cdot\frac{(-1)^n}{n}\cdot$$

$$\frac{1}{\left(1+\frac{n^2\pi^2}{L^2}\,\frac{\Omega_1}{\Omega_2}\right)^{3/2}}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\left\{1+\frac{\Omega_1}{\Omega_2}\left[\frac{(2\nu+1)\pi}{2L}\right]^2\right\}^{3/2}}\cdot$$

$$\cdot \int_{\tau=0}^{\tau=t} \sin \frac{\tau \sqrt{\frac{\Omega_3}{\Omega_2}}}{\left[1 + \frac{4L^2}{\pi^2} \frac{\frac{\Omega_2}{\Omega_1}}{(2\nu + 1)^2}\right]^{1/2}} \cdot \sin \frac{n\pi}{L} \frac{\sqrt{\frac{\Omega_3}{\Omega_2}} (t-\tau)}{\left(1 + \frac{\pi^2 n^2}{L^2} \frac{\Omega_1}{\Omega_2}\right)^{1/2}} d\tau \tag{171}$$

For $n = 0$:

$$\mathcal{L}^{-1}\left\{\frac{\phi(p)\left(\Omega_3 + \Omega_1 p^2\right)}{\Omega_2 p^3}\right\} = -\frac{P\epsilon z}{KL}\left(\frac{\Omega_3}{\Omega_2}\right)^{3/2} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\left\{1 + \frac{\Omega_1}{\Omega_2}\left[\frac{(2\nu + 1)\pi}{2L}\right]^2\right\}^{3/2}} \cdot$$

$$\int_0^t (t-\tau) \sin \frac{\sqrt{\frac{\Omega_3}{\Omega_1}}\, \tau}{\left[1 + \frac{4L^2}{\pi^2} \frac{\frac{\Omega_2}{\Omega_1}}{(2\nu + 1)^2}\right]^{1/2}} d\tau \tag{172}$$

We put, for simplicity,

$$c = \frac{n\pi \sqrt{\frac{\Omega_3}{\Omega_2}}}{L \sqrt{1 + \frac{n^2\pi^2}{L^2} \frac{\Omega_1}{\Omega_2}}} \quad , \quad a = \frac{\sqrt{\frac{\Omega_3}{\Omega_2}}}{\sqrt{1 + \frac{4L^2}{\pi^2} \frac{\frac{\Omega_2}{\Omega_1}}{(2\nu + 1)^2}}} \tag{173}$$

and solve the integrals

$$\int_0^t \sin a\tau \sin c(t-\tau)\, d\tau = \frac{ac}{a^2 - c^2}\left(\frac{1}{c} \sin ct - \frac{1}{a} \sin at\right) \tag{174}$$

$$\int_0^t (t-\tau) \sin a\tau d\tau = \frac{1}{a}\left(t - \frac{1}{a} \sin at\right) \tag{175}$$

Substituting equations (174) and (175) into equations (171) and (172), and taking into account equation (173), we finally obtain the following from equation (167):

$$w(z,t) = -\frac{1}{L}\frac{P\varepsilon}{K}\frac{\Omega_3}{\Omega_2}\left\{t\left[1 - \frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\left[1 + \frac{\Omega_1}{\Omega_2}\left(\frac{\pi\ (2\nu + 1)^2}{2L}\right)^2\right](2\nu + 1)}\right] + \right.$$

$$\frac{8L}{\pi^2}\sqrt{\frac{\Omega_2}{\Omega_3}} \cdot \sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{(2\nu + 1)^2\left[1 + \frac{\Omega_1}{\Omega_2}\left(\frac{(2\nu + 1)\pi}{2L}\right)^2\right]} \cdot$$

$$\left.\sin\frac{\sqrt{\frac{\Omega_3}{\Omega_2}}\ t}{\left[1 + \frac{4L^2}{\pi^2}\frac{\frac{\Omega_2}{\Omega_1}}{(2\nu + 1)^2}\right]^{\frac{1}{2}}}\right\} - \frac{2P\varepsilon}{K\pi}\sqrt{\frac{\Omega_3}{\Omega_1}} \cdot$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n \cos\frac{n\pi z}{L}}{n} \cdot \sin\frac{\sqrt{\frac{\Omega_3}{\Omega_2}}\ t}{\sqrt{1 + \frac{4L^2}{\pi^2}\frac{\frac{\Omega_2}{\Omega_1}}{(2n)^2}}} \cdot$$

$$\cdot \left\{ \frac{1}{\left[1 + \frac{n^2\pi^2}{L^2}\frac{\Omega_1}{\Omega_2}\right]^{3/2}} - 2\left(\frac{2L}{\pi}\right)^3 \sqrt{\frac{\Omega_2}{\Omega_1}} \cdot \frac{1}{(2n)^2\left(1+\frac{(2n)^2}{4L^2}\frac{\Omega_1}{\Omega_2}\right)^{\frac{1}{2}}} \cdot \right.$$

$$\left. \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\left[2(\nu-n)+1\right]\left[2(\nu+n)+1\right](2\nu+1)\left[1+\frac{\Omega_1}{\Omega_2}\left(\frac{(2\nu+1)\pi}{2L}\right)^2\right]} \right\} -$$

$$\frac{2P\varepsilon L}{\pi K}\left(\frac{2L}{\pi}\right)^3 \frac{\sqrt{\Omega_2\Omega_3}}{\Omega_1} \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{2n^2} \frac{\cos\frac{n\pi z}{L}}{1+\frac{(2n)^2}{4L^2}\frac{\Omega_1}{\Omega_2}} \cdot$$

$$\sum_{\nu=0}^{\infty} \frac{(-1)^\nu \cdot \dfrac{\sqrt{\frac{\Omega_3}{\Omega_1}}\, t}{\left[1+\frac{4L^2}{\pi^2}\dfrac{\frac{\Omega_2}{\Omega_1}}{(2\nu+1)^2}\right]^{\frac{1}{2}}}}{\left[2(\nu-n)+1\right]\left[2(\nu+n)+1\right](2\nu+1)^2\left[1+\frac{\Omega_1}{\Omega_2}\left(\frac{(2\nu+1)\pi}{2L}\right)^2\right]^{\frac{1}{2}}} \qquad (176)$$

In the developing of this expression the equation (142) has been used, because it corresponds to the inverse of the first term of the right hand of equation (167).

Now we will consider the fixed-free composite under a sudden load (Figure 5).

Thus, the boundary condition at the free end is

$$\frac{\partial w}{\partial z}(L,t) = \begin{cases} 0 & \text{for} \quad t \leq 0 \\ \dfrac{P}{K} & \text{for} \quad 0 < t < \infty \end{cases} \tag{177}$$

instead of the condition given in equation (145).

The case of sudden load will be solved as the limit when $\varepsilon \to \infty$ of the impact load case. Thus, the equations (159) and (168) become

$$\phi(p) = \frac{P}{K} \frac{1}{\operatorname{ch} \dfrac{L\sqrt{\Omega_2}\, p}{\sqrt{\Omega_3 + p^2 \Omega_1}}} \tag{178}$$

$$\mathcal{L}^{-1}\left\{ \frac{\phi(p)}{\frac{P}{K}} \cdot \frac{\frac{P}{K}(\Omega_2 + \Omega_1 p^2)}{p\left[\Omega_3 \dfrac{n^2\pi^2}{L^2} + \left(\Omega_1 \dfrac{n^2\pi^2}{L^2} + \Omega_2\right) p^2\right]} \right\} = \int_0^t f(\tau)\, \bar{g}_n(t-\tau)\, d\tau \tag{179}$$

with

For $n \neq 0$:

$$\bar{g}_n(t-\tau) = \frac{PL^2}{K\pi^2}\,\frac{(-1)^n}{n^2}\left\{1 - \frac{\cos \dfrac{n\pi\sqrt{\dfrac{\Omega_3}{\Omega_2}}\,(t-\tau)}{L\sqrt{1+\dfrac{\pi^2 n^2}{L^2}\,\dfrac{\Omega_1}{\Omega_2}}}}{1+\dfrac{n^2\pi^2}{L^2}\,\dfrac{\Omega_1}{\Omega_2}}\right\} \qquad (180)$$

for $n = 0$:

$$\bar{g}_0(t-\tau) = \frac{P}{K}\,\frac{\Omega_3}{\Omega_2}\left[\frac{\Omega_1}{\Omega_3} + \frac{(t-\tau)^2}{2}\right] \qquad (181)$$

inverse Laplace transform of the second factor in the left-hand of equation (179).

On the other hand, the inverse Laplace transformation of $\phi(p)$ given in equation (178), is

$$f(\tau) = \frac{2}{L}\sqrt{\frac{\Omega_3}{\Omega_2}}\sum_{\nu=0}^{\infty}(-1)^{\nu}\,\frac{\sin\dfrac{\sqrt{\dfrac{\Omega_3}{\Omega_1}}\,\tau}{\left[1+\dfrac{4L^2}{\pi^2}\,\dfrac{\dfrac{\Omega_2}{\Omega_1}}{(2\nu+1)^2}\right]^{1/2}}}{\left[1+\dfrac{\Omega_1}{\Omega_2}\,\dfrac{(2\nu+1)^2\pi^2}{4L^2}\right]^{3/2}} \qquad (182)$$

With the equations (180), (181), and (182) the equation (179) is expressed in the following form

For $n \neq 0$:

$$\bar{w}_n(t) = \mathcal{L}^{-1}\left\{\frac{\phi(p)\left(\Omega_3 + \Omega_1 p^2\right)}{p\left[\Omega_3 \frac{n^2\pi^2}{L^2} + \left(\Omega_1 \frac{n^2\pi^2}{L^2} + \Omega_2\right) p^2\right]}\right\}_{\epsilon=\infty}$$

$$= (-1)^n \frac{2PL}{n^2\pi^2 K}\sqrt{\frac{\Omega_3}{\Omega_2}} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\left[1 + \frac{\Omega_1}{\Omega_2}\frac{(2\nu+1)^2\pi^2}{4L^2}\right]^{3/2}} \cdot$$

$$\int_0^t \left(1 - \frac{\cos \dfrac{n\pi\sqrt{\frac{\Omega_3}{\Omega_2}}(t-\tau)}{L\sqrt{1 + \frac{n^2\pi^2}{L^2}\frac{\Omega_1}{\Omega_2}}}}{1 + \frac{n^2\pi^2}{L^2}\frac{\Omega_1}{\Omega_2}}\right) \sin \frac{(2\nu+1)\pi\sqrt{\frac{\Omega_3}{\Omega_2}}\,\tau}{2L\sqrt{1 + \frac{\pi^2(2\nu+1)^2}{4L^2}\frac{\Omega_1}{\Omega_2}}}\, d\tau \qquad (183)$$

For $n = 0$:

$$\bar{w}_0(t) = \mathcal{L}^{-1}\left\{\frac{\phi(p)\left(\Omega_3 + \Omega_1 p^2\right)}{\Omega_2 p^3}\right\}_{\epsilon=\infty} = \frac{2P}{KL}\left(\frac{\Omega_3}{\Omega_2}\right)^{3/2} \cdot$$

$$\sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\left[1 + \frac{\Omega_1}{\Omega_2}\frac{(2\nu+1)^2\pi^2}{4L^2}\right]^{3/2}} \int_0^t \left(\frac{\Omega_1}{\Omega_3} + \frac{t^2}{2} - t\tau + \frac{\tau^2}{2}\right) \cdot$$

$$\sin \frac{(2\nu+1)\,\pi\sqrt{\frac{\Omega_3}{\Omega_2}}\,\tau}{2L\sqrt{1 + \frac{\pi^2(2\nu+1)^2}{4L^2}\frac{\Omega_1}{\Omega_2}}}\, d\tau \qquad (184)$$

We put, for the sake of brevity,

$$c = \frac{n\pi}{L} \frac{\sqrt{\frac{\Omega_3}{\Omega_2}}}{\sqrt{1 + \frac{n^2\pi^2}{L^2}\frac{\Omega_1}{\Omega_2}}} \tag{185}$$

$$a = \frac{(2\nu + 1)\pi}{2L} \frac{\sqrt{\frac{\Omega_3}{\Omega_2}}}{\sqrt{1 + \frac{\pi^2(2\nu + 1)^2}{4L^2}\frac{\Omega_1}{\Omega_2}}} \tag{186}$$

On the other hand, we have

$$\int_0^t \sin a\tau d\tau = \frac{1}{a}(1 - \cos at) \tag{187}$$

$$\int_0^t \sin\left\{ct + (a-c)\tau\right\} d\tau - \int_0^t \sin\left\{ct - (a+c)\tau\right\} d\tau = \frac{a}{a^2 - c^2}(\cos ct - \cos at) \tag{188}$$

$$\int_0^t (t-\tau)^2 \sin a\tau d\tau = \frac{t^2}{a} - \frac{2}{a^3}(1 - \cos at) \tag{189}$$

By substitution of equations (187), (188), and (189) into equations (183) and (184) and considering equations (185) and (186), after some algebraic manipulations, we obtain

$$\bar{w}_n(t) = (-1)^n \frac{4PL^2}{\pi^3 K} \left\{ \left[-\cos\left(\frac{n\pi t}{L} \frac{\sqrt{\frac{\Omega_3}{\Omega_2}}}{\sqrt{1 + \frac{\pi^2 n^2}{L^2} \frac{\Omega_1}{\Omega_2}}} \right) \cdot \right. \right.$$

$$\sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}(2\nu+1)}{\left(1 + \frac{\Omega_1}{\Omega_2} \frac{(2\nu+1)^2 \pi^2}{4L^2}\right) \left[(2\nu+1)^2 - 4n^2\right]} +$$

$$\left. \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{(2\nu+1)\left(1 + \frac{\Omega_1}{\Omega_2} \frac{(2\nu+1)^2 \pi^2}{4L^2}\right)} \right] \frac{1}{n^2} +$$

$$\left. 4 \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu} \cos \dfrac{(2\nu+1)\pi \sqrt{\frac{\Omega_3}{\Omega_2}}\, t}{2L \sqrt{1 + \frac{\pi^2 (2\nu+1)^2}{4L^2} \frac{\Omega_1}{\Omega_2}}}}{(2\nu+1)\left[(2\nu+1)^2 - 4n^2\right] \left[1 + \frac{\Omega_1}{\Omega_2} \frac{(2\nu+1)^2 \pi^2}{4L^2}\right]} \right\} \qquad (190)$$

$$\overline{w}_0(t) = \frac{2P}{KL}\left(\frac{\Omega_3}{\Omega_2}\right)^{3/2}\left\{\frac{2L}{\pi}\frac{\Omega_1}{\Omega_3}\sqrt{\frac{\Omega_2}{\Omega_3}}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{(2\nu+1)\left(1+\frac{\Omega_1}{\Omega_2}\frac{(2\nu+1)^2\pi^2}{4L^2}\right)} + \right.$$

$$\frac{8L^3}{\pi^3}\left(\frac{\Omega_2}{\Omega_3}\right)^{3/2}\sum_{\nu=0}^{\infty}\frac{\cos\left[\dfrac{(2\nu+1)\pi\sqrt{\dfrac{\Omega_3}{\Omega_2}}\,t}{2\ell\sqrt{1+\dfrac{\pi^2(2\nu+1)^2}{4L^2}\dfrac{\Omega_1}{\Omega_2}}}\right]-1}{(2\nu+1)^3\left[1+\dfrac{(2\nu+1)^2\pi^2}{4L^2}\dfrac{\Omega_1}{\Omega_2}\right]} +$$

$$\left.\frac{2L}{\pi}\left(\frac{\Omega_2}{\Omega_3}\right)^{1/2}\left(\frac{t^2}{2}-\frac{\Omega_2}{\Omega_1}\right)\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{(2\nu+1)\left(1+\frac{\Omega_1}{\Omega_2}\frac{(2\nu+1)^2\pi^2}{4L^2}\right)}\right\} \tag{191}$$

But, taking into account that

$$\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{(2\nu+1)\left[1+\dfrac{\Omega_1}{\Omega_2}\dfrac{\pi^2}{4L^2}(2\nu+1)^2\right]} = \frac{\pi}{4}-\frac{\pi}{2}\,\frac{\mathrm{sh}\left(L\sqrt{\dfrac{\Omega_2}{\Omega_1}}\right)}{\mathrm{sh}\left(2L\sqrt{\dfrac{\Omega_2}{\Omega_1}}\right)} \tag{192}$$

$$\sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}(2\nu+1)}{\left[1 + \frac{\Omega_1}{\Omega_2}\frac{\pi^2}{4L^2}(2\nu+1)^2\right]\left[-4n^2 + (2\nu+1)^2\right]}$$

$$= \frac{L^2}{4\pi}\cdot\frac{\Omega_2}{\Omega_1}\cdot\frac{(-1)^n - 2\,\dfrac{\mathrm{sh}\left(L\sqrt{\dfrac{\Omega_2}{\Omega_1}}\right)}{\mathrm{sh}\left(2L\sqrt{\dfrac{\Omega_2}{\Omega_1}}\right)}}{n^2 + \dfrac{L^2}{\pi^2}\dfrac{\Omega_2}{\Omega_1}} \qquad (193)$$

Thus, by using equations (192) and (193), equations (190) and (191) become

$$\bar{w}_n(t) = (-1)^n \frac{4PL^2}{\pi^3 K}\left\{\frac{1}{n^2}\left[\frac{\pi}{4} - \frac{\pi}{2}\,\frac{\mathrm{sh}\left(L\sqrt{\dfrac{\Omega_2}{\Omega_1}}\right)}{\mathrm{sh}\left(2L\sqrt{\dfrac{\Omega_2}{\Omega_1}}\right)} + \right.\right.$$

$$\left.+ \frac{L^2}{4\pi}\frac{\Omega_2}{\Omega_1}\,\frac{(-1)^n - 2\,\dfrac{\mathrm{sh}\left(L\sqrt{\dfrac{\Omega_2}{\Omega_1}}\right)}{\mathrm{sh}\left(2L\sqrt{\dfrac{\Omega_2}{\Omega_1}}\right)}}{\dfrac{L^2}{\pi^2}\dfrac{\Omega_2}{\Omega_1} + n^2}\cos\left(\frac{n\pi}{L}\,\frac{\sqrt{\dfrac{\Omega_3}{\Omega_2}}}{\sqrt{1 + \dfrac{n^2\pi^2}{L^2}\dfrac{\Omega_1}{\Omega_2}}}\,t\right)\right] +$$

$$
+\ 4 \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu} \cos\left(\dfrac{(2\nu+1)\,\pi \sqrt{\dfrac{\Omega_3}{\Omega_2}}}{2L \sqrt{1 + \dfrac{\pi^2}{4L^2} (2\nu+1)^2 \dfrac{\Omega_1}{\Omega_2}}}\, t \right)}{(2\nu+1)\left[(2\nu+1)^2 - 4n^2\right]\left[1 + \dfrac{\Omega_1}{\Omega_2} \dfrac{\pi^2}{4L^2} (2\nu+1)^2\right]} \Bigg\} \qquad (194)
$$

$$
\bar{w}_0(t) = \frac{4P}{K\pi} \Bigg\{ \frac{\Omega_3}{\Omega_2} \frac{t^2}{2} \left[\frac{\pi}{4} - \frac{\pi}{2} \frac{\mathrm{sh}\left(L \sqrt{\dfrac{\Omega_2}{\Omega_1}} \right)}{\mathrm{sh}\left(2L \sqrt{\dfrac{\Omega_2}{\Omega_1}} \right)} \right] - \frac{4L^2}{\pi^2} \cdot
$$

$$
\sum_{\nu=0}^{\infty} \frac{1 - \cos\left[\dfrac{(2\nu+1)\pi}{2L} \cdot \dfrac{\sqrt{\dfrac{\Omega_3}{\Omega_2}}}{\sqrt{1 + \dfrac{\pi^2}{4L^2} (2\nu+1)^2 \dfrac{\Omega_1}{\Omega_2}}} \right]}{(2\nu+1)^3 \left[1 + \dfrac{(2\nu+1)^2 \pi^2}{4L^2} \dfrac{\Omega_1}{\Omega_2} \right]} \Bigg\} \qquad (195)
$$

Finally, applying the finite cosine Fourier inversion, we find the displacement function for the sudden load at the fixed-free composite:

$$w(z,t) = \frac{1}{L} w_0(t) + \frac{2}{L}\sum_{n=1}^{\infty} w_n(t) \cos \frac{n\pi z}{L}$$

$$= \frac{4P\ell}{\pi K}\left\{ \frac{\pi}{L^2}\frac{\Omega_3}{\Omega_2}\left[\frac{1}{2} - \frac{\mathrm{sh}\left(L\sqrt{\frac{\Omega_2}{\Omega_1}}\right)}{\mathrm{sh}\left(2L\sqrt{\frac{\Omega_2}{\Omega_1}}\right)}\right]\frac{t^2}{4} - \frac{4}{\pi^2} \cdot \right.$$

$$\left. \sum_{\nu=0}^{\infty} \frac{1 - \cos\left[\frac{(2\nu+1)\pi}{2L}\sqrt{\frac{\Omega_3}{\Omega_2}}\left(1 + \frac{\pi^2}{4L^2}\frac{\Omega_1}{\Omega_2}(2\nu+1)^2\right)^{-1/2} t\right]}{(2\nu+1)^3\left[1 + \frac{(2\nu+1)^2\pi^2}{4L^2}\frac{\Omega_1}{\Omega_2}\right]}\right\} +$$

$$\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left\{\frac{\pi}{4}\left(1 - 2\,\frac{\mathrm{sh}\left(L\sqrt{\frac{\Omega_2}{\Omega_1}}\right)}{\mathrm{sh}\left(2L\sqrt{\frac{\Omega_2}{\Omega_1}}\right)}\right) + \right.$$

$$\pi\frac{\Omega_2}{\Omega_1}\,\frac{(-1)^n - 2\,\frac{\mathrm{sh}\left(L\sqrt{\frac{\Omega_2}{\Omega_1}}\right)}{\mathrm{sh}\left(2L\sqrt{\frac{\Omega_2}{\Omega_1}}\right)}}{4\frac{\Omega_2}{\Omega_1} + \left(\frac{2\pi n}{L}\right)^2}\cos\left(\frac{n\pi}{L}\cdot\frac{\sqrt{\frac{\Omega_3}{\Omega_2}}}{\sqrt{1 + \frac{\pi^2 n^2}{L^2}\frac{\Omega_1}{\Omega_2}}}\,t\right) +$$

$$+\,4\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}\cos\left[\frac{(2\nu+1)\pi}{2L}\sqrt{\frac{\Omega_3}{\Omega_2}}\left(1+\frac{\pi^2(2\nu+1)^2}{4L^2}\frac{\Omega_1}{\Omega_2}\right)^{-1/2}t\right]}{(2\nu+1)\left[\left(\frac{2\nu+1}{2n}\right)^2-1\right]\left[1+\frac{\pi^2}{4L^2}\frac{\Omega_1}{\Omega_2}(2\nu+1)^2\right]}\Bigg\}\cos\frac{n\pi z}{L} \tag{196}$$

In the following text, a composite of semi-infinite length subjected to both sudden and impulsive loads will be considered.

The case of impact for a composite of infinite length can be developed as the limit of the free-free case with impulsive load when the length tends to infinity. Taking

$$\Delta\xi = \frac{\pi}{L}\quad,\quad \xi_n = \frac{n\pi}{L}\quad,\quad L-z = \zeta \tag{197}$$

where ζ is the distance from the impacted end. With equations (197), equation (142) becomes

$$w(\zeta,t) = \frac{I}{K}\frac{\Omega_3}{\Omega_2}\frac{\Delta\xi}{\pi}t + \frac{2I}{\pi K}\sqrt{\frac{\Omega_3}{\Omega_2}}\,\cdot$$

$$\sum_{n=1}^{\infty}\frac{\cos(\xi_n\zeta)\sin\left(\xi_n\frac{at}{\sqrt{1+\xi_n^2b^2}}\right)}{\xi_n\left(1+\xi_n^2b^2\right)^{3/2}}\Delta\xi \tag{198}$$

with

$$b = \sqrt{\frac{\Omega_1}{\Omega_2}}\quad,\quad a = \sqrt{\frac{\Omega_3}{\Omega_2}} \tag{199}$$

Taking the limit of expression (198) when $\Delta\xi \to 0$ and $n \to \infty$, we find

$$w(\zeta,t) = -\frac{2}{\pi}\frac{I}{K}\sqrt{\frac{\Omega_3}{\Omega_2}}\int_0^\infty \frac{\cos(\xi,\zeta)\sin\left(\xi\frac{at}{\sqrt{1+\xi^2b^2}}\right)d\xi}{\xi\,(1+\xi^2b^2)^{3/2}} \qquad (200)$$

In Appendix V, details on the evaluation of the integral in equation (200) are given. The final result is

$$w(\zeta,t) = -\frac{1}{2}\frac{I}{K}\sqrt{\frac{\Omega_3}{\Omega_2}}\cdot e^{-\frac{\zeta}{b}} \cdot$$

$$\sum_{\nu=0}^{\infty}\frac{(-i)^{\nu}\,\nu!}{(2\nu+1)!}\left(\frac{at}{2b}\right)^{2\nu+1}\left(\frac{z}{b}\right)^{\nu}\sum_{\mu=0}^{\nu}\frac{(-1)^{\mu}\left(\frac{z}{b}\right)^{\mu}}{\mu!\,(\nu-\mu)!\,(\nu+\mu+3)!} \qquad (201)$$

$$\sum_{k=-3}^{\infty}\left(\frac{b}{2z}\right)^{k}\frac{(\nu+\mu+k+6)!}{(3+k)!\,(\nu+\mu-k)!}$$

The case of sudden load applied on a composite of infinite length will now be considered. Taking the limit for $L \to \infty$ at formula (136) corresponding to the case of sudden load for finite length, we obtain

$$w(z,t) = \frac{2P}{K\pi}\sum_{n=1}^{\infty}\frac{\pi}{L}(-1)^n\frac{\cos\frac{n\pi z}{L}}{\left(\frac{\pi n}{L}\right)^2}\left\{1-\frac{\cos\frac{n\pi}{L}\frac{\sqrt{\frac{\Omega_3}{\Omega_2}}\,t}{\sqrt{1+\left(\frac{n\pi}{L}\right)^2\frac{\Omega_1}{\Omega_2}}}}{1+\left(\frac{n\pi}{L}\right)^2\frac{\Omega_1}{\Omega_2}}\right\} \qquad (202)$$

or, using equations (209) and doing $\Delta\xi \to 0$ and $n \to \infty$

$$w(\zeta,t) = \frac{2P}{K\pi}\int_0^\infty \frac{\cos \xi\zeta}{\xi^2}\left[1 - \frac{\cos\left(\dfrac{a\xi t}{\sqrt{1+\xi^2 b^2}}\right)}{1+\xi^2 b^2}\right] d\xi \tag{203}$$

This integral can be evaluated using a similar method that is employed in Appendix VI, or by direct numerical integration.

CONSTANTS OF THE FUNDAMENTAL DIFFERENTIAL EQUATION FOR THE HEXAGONAL ARRANGEMENT OF FIBERS

In this section, the constants Ω_1, Ω_2, and Ω_3 of the fundamental differential equation are computed by using the boundary conditions that correspond to the hexagonal arrangement of the fibers, instead of the assumed symmetry of revolution made before.

According to the method indicated in Section II, the solution of a plane strain problem must be found. Then the use of the Airy function, $\phi(r,\theta)$ defined by

$$\left.\begin{aligned} \sigma_r &= \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\phi\theta^2} \\ \sigma_\theta &= \frac{\partial^2\phi}{\partial r^2} \\ \tau_{r\theta} &= \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} \end{aligned}\right\} \tag{204}$$

is appropriate.

Because of the symmetrical arrangement (see Figure 11), it is necessary to consider only the shaded area. The boundary conditions for the element of Figure 12 are

$$\left.\begin{aligned}
\left(\sigma^{I}_{rr}\right)_{r=a} &= \left(\sigma^{II}_{rr}\right)_{r=a} \quad ; & \left(\sigma^{I}_{r\theta}\right)_{r=a} &= \left(\sigma^{II}_{r\theta}\right)_{r=a} \\
\left(u^{I}_{r}\right)_{r=a} &= \left(u^{II}_{r}\right)_{r=a} \quad ; & \left(u^{I}_{\theta}\right)_{r=a} &= \left(u^{II}_{\theta}\right)_{r=a} \\
u_{n}\Big)_{r=\frac{b}{\cos\left(\frac{\pi}{6}-\theta\right)}} &= 0 \quad ; & \left(\sigma_{nt}\right)_{r=\frac{b}{\cos\left(\frac{\pi}{6}-\theta\right)}} &= 0
\end{aligned}\right\} \qquad (205)$$

The first four conditions correspond to the interface, and the last two correspond to the straight line AB (see Figure 12), referred to the unit vectors $\bar{n}$ and $\bar{t}$. In the following, instead of r, θ, z, is sometimes written 1, 2, 3, correspondingly.

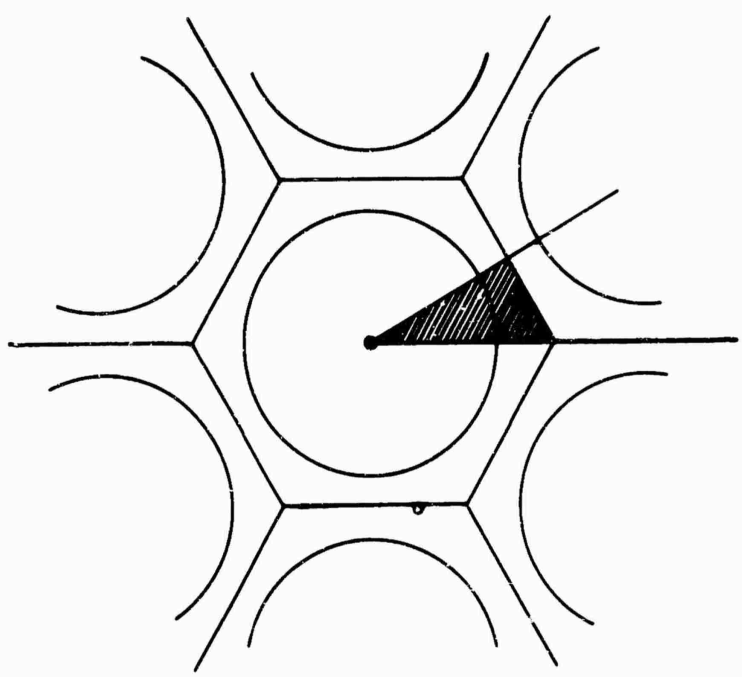

Figure 11. Hexagonal Element

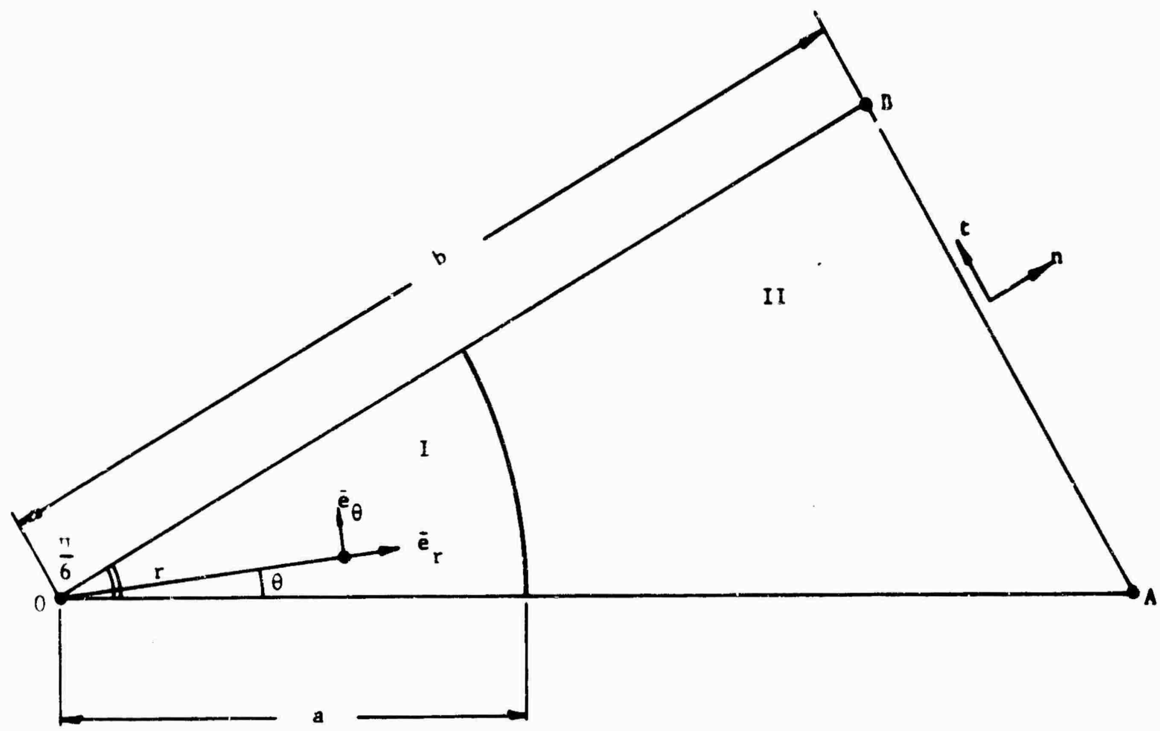

Figure 12. Geometry Description

We adopt as the Airy function

$$\phi = B_0 \ln r + C_0 r^2 + \sum_{n=6,12}^{\infty} (A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2}) \cos n\theta \qquad (206)$$

By substituting equation (206) into equations (204), the following stresses are obtained:

$$\left.\begin{aligned}
\sigma_{11} &= B_0 r^{-2} + 2C_0 - \sum_n \Big[n(n-1)\, A_n r^{n-2} + n(n+1)\, B_n r^{-n-2} + \\
&\qquad (n+1)(n-2)\, C_n r^n + (n-1)(n+2)\, D_n r^{-n} \Big] \cos n\theta \\
\sigma_{22} &= -B_0 r^{-2} + 2C_0 + \sum_n \Big[n(n-1)\, A_n r^{n-2} + n(n+1)\, B_n r^{-n-2} + \\
&\qquad (n+1)(n+2)\, C_n r^n + (n-1)(n-2)\, D_n r^{-n} \Big] \cos n\theta \\
\sigma_{12} &= \sum_n \Big[n(n-1)\, A_n r^{n-2} - n(n+1)\, B_n r^{-n-2} + n(n+1)\, C_n r^n - \\
&\qquad n(n-1)\, D_n r^{-n} \Big] \sin n\theta
\end{aligned}\right\} \quad (207)$$

By substituting equations (207) into the expressions of the generalized Hooke law and then integrating, we obtain the displacements

$$\left.\begin{aligned}
u_r &= \frac{1+\nu}{E} \Big\{ - B_0 r^{-1} + 2(1-2\nu)\, C_0 r - \sum_n^{\infty} \Big[n\, A_n r^{n-1} + \\
&\qquad (n-2+4\nu)\, C_n r^{n+1} - (n+2-4\nu)\, D_n r^{-n+1} \Big] \cos n\theta \Big\} \\
u_\theta &= \frac{1+\nu}{E} \sum_n \Big[n\, A_n r^{n-1} + n\, B_n r^{-n-1} + (n+4-4\nu)\, C_n r^{n+1} + \\
&\qquad (n-4+4\nu)\, D_n r^{-n+1} \Big] \sin n\theta
\end{aligned}\right\} \quad (208)$$

To satisfy the symmetry conditions, we must take $n=6,12,18,\ldots$ in equations (207) and (208). However, the displacements and stresses for $r=0$ must be finite. Then, for the fiber,

$$B_o^I = B_n^I = D_n^I = 0 \tag{209}$$

Thus, considering equations (207), (208), and (209), the stresses and displacements in the fiber (Index I) are

$$\left.\begin{aligned}
\sigma_{11}^I &= 2C_o^I - \sum_{n=6,12,\ldots} \left[n(n-1)\, A_n^I r^{n-2} + (n+1)(n-2)\, C_n^I r^n \right] \cos n\theta \\
\sigma_{22}^I &= 2C_o^I + \sum_{n=6,12,\ldots} \left[n(n-1)\, A_n^I r^{n-2} + (n+1)(n+2)\, C_n^I r^n \right] \cos n\theta \\
\sigma_{12}^I &= \sum_{n=6,12,\ldots} \left[n(n-1)\, A_n^I r^{n-2} + n(n+1)\, C_n^I r^n \right] \sin n\theta
\end{aligned}\right\} \tag{210}$$

$$\left.\begin{aligned}
u_r^I &= \frac{1+\nu_I}{E_I} \Bigg\{ 2\left(1 - 2\nu_I^I\right) C_o^I r - \\
&\qquad \sum_{n=6,12\ldots} \left[n\, A_n^I\, r^{n-1} + \left(n - 2 + 4\nu_I^I\right) C_n^I\, r^{n+1} \right] \cos n\theta \Bigg\} + \nu_I\, \epsilon_{33}\, r \\
u_\theta^I &= \frac{1+\nu_I}{E_I} \sum_{n=6,12,\ldots} \left[n\, A_n^I\, r^{n-1} + \left(n + 4 - 4\nu^I\right) C_n^I\, r^{n+1} \right] \sin n\theta
\end{aligned}\right\} \tag{211}$$

and, for the matrix (index II), are

$$
\left.
\begin{aligned}
\sigma_{11}^{II} &= B_o^{II}\, r^{-2} + 2C_o^{II} - \sum_{n=6,12,\ldots} \Big[n(n-1)\, A_n^{II}\, r^{n-2} + n(n+1) \cdot \\
&\qquad B_n^{II}\, r^{-n-2} + (n+1)(n-2)\, C_n^{II}\, r^{n} + (n-1)(n+2) D_n^{II}\, r^{-n} \Big] \cos n\theta \\
\sigma_{22}^{II} &= -B_o^{II} r^{-2} + 2C_o^{II} + \sum_{n=6,12,\ldots} \Big[n(n-1)\, A_n^{II} r^{n-2} + n(n+1) \cdot \\
&\qquad B_n^{II}\, r^{-n-2} + (n+1)(n+2)\, C_n^{II}\, r^{n} + (n-1)(n-2) D_n^{II}\, r^{-n} \Big] \cos n\theta \\
\sigma_{12}^{II} &= \sum_{n=6,12\ldots} \Big[n(n-1) A_n^{II}\, r^{n-2} - n(n+1) B_n^{II} r^{-n-2} + \\
&\qquad n(n+1) C_n^{II}\, r^{n} - n(n-1) D_n^{II}\, r^{-n} \Big] \sin n\theta
\end{aligned}
\right\} \quad (212)
$$

$$
\left.
\begin{aligned}
u_r^{II} &= \frac{1 + \nu_{II}}{E_{II}} \cdot \\
&\left\{ -B_o^{II} r^{-1} + 2\left(1 - 2\nu_{II}\right) C_o^{II} r - \sum_{n=6} \left[n A_n^{II} r^{n-1} - nB_n^{II} r^{-n-1} + \right.\right. \\
&\left.\left. \left(n - 2 + 4\nu_{II}\right) C_n^{II} r^{n+1} - \left(n + 2 - 4\nu_{II}\right) D_n^{II} r^{-n+1} \right] \cos n\theta \right\} + \nu_{II}\, \epsilon_{33}\, r \\
u_\theta^{II} &= \frac{1 + \nu_{II}}{E_{II}} \sum_n \left[n A_n^{II} r^{n-1} + nB_n^{II} r^{-n-1} + \left(n + 4 - 4\nu_{II}\right) \cdot \right. \\
&\left. C_n^{II} r^{n+1} + \left(n - 4 + 4\nu_{II}\right) D_n^{II} r^{-n+1} \right] \sin n\theta
\end{aligned}
\right\} \quad (213)
$$

The "point matching" method may be used to find the constants that appear in equations (210) through (213). The nine constants A_n^I , C_n^I , C_o^I , A_n^{II} , B_n^{II} , C_n^{II} , D_n^{II} , B_o^{II} , and C_o^{II} represent

$$p = 3 + 6n$$

unknowns.

In each point of Type (a) in Figure 13, it is necessary to take the four boundary conditions given by the first four equations (205). For each point of Type (b), we have the two boundary conditions given by the last two equations (205).

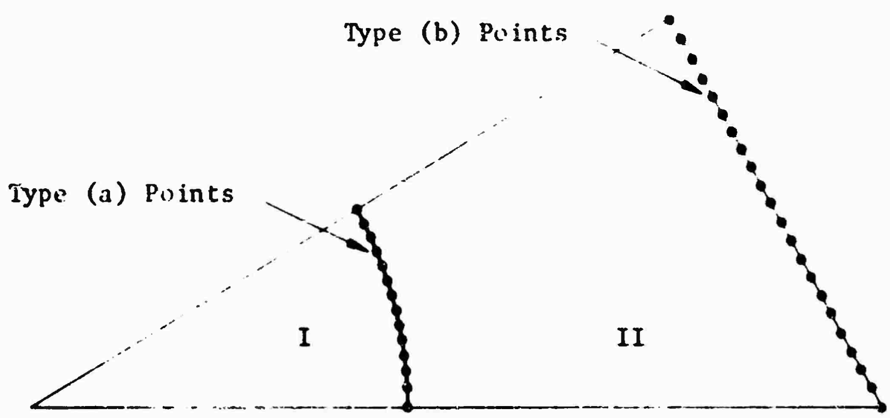

Figure 13. Distribution of Matching Points

Taking, for example, 8 points (a), 12 points (b), and $n = 4$, we arrive at a system of 56 linear algebraic equations with only 27 constants unknown. Solving this system by the least squares and substituting the identified constants in equations (210) through (213), we have the stresses and displacements as functions of the imposed plane strain ϵ_{33}.

Now we must compute the strain (potential) W and kinetic T energies. The total energies are given by

$$W = \int_0^{\pi/6}\int_0^a \overline{W}_{o_I}\, r dr\, d\theta + \int_0^{\pi/6} d\theta \int_a^{\frac{b}{\cos\left(\frac{\pi}{6}-\theta\right)}} \overline{W}_{o_{II}}\, r dr +$$

$$\int_0^{\pi/6} d\theta \int_0^a \overline{W}_{1_I}\, r dr + \int_0^{\pi/6} d\theta \int_a^{\frac{b}{\cos\left(\frac{\pi}{6}-\theta\right)}} \overline{W}_{1_{II}}\, r dr \tag{214}$$

$$T = \int_0^{\pi/6} d\theta \int_0^a \rho_I \left[\left(\frac{\partial u^I}{\partial t}\right)^2 + \left(\frac{\partial v^I}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] r dr +$$

$$\int_0^{\pi/6} d\theta \int_a^{\frac{b}{\cos\left(\frac{\pi}{6}-\theta\right)}} \rho_{II} \left[\left(\frac{\partial u^{II}}{\partial t}\right)^2 + \left(\frac{\partial v^{II}}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] r dr \tag{215}$$

where

$$\bar{W}_{o_I} = \frac{E_I}{2}\left(\frac{\partial w}{\partial z}\right)^2 \quad , \quad \bar{W}_{o_{II}} = \frac{E_{II}}{2}\left(\frac{\partial w}{\partial z}\right)^2 \tag{216}$$

$$\left.\begin{aligned}
\bar{W}_{1_I} &= \frac{1}{2}\left[\sigma_{11}^{I}\frac{\partial u^I}{\partial r} + \sigma_{22}^{I}\left(\frac{u^I}{r} + \frac{1}{r}\frac{\partial v^I}{\partial \theta}\right) + \sigma_{12}^{I}\left(\frac{\partial v^I}{\partial r} - \frac{v^I}{r} + \frac{1}{r}\frac{\partial u^I}{\partial \theta}\right)\right] \\
\bar{W}_{1_{II}} &= \frac{1}{2}\left[\sigma_{11}^{II}\frac{\partial u^{II}}{\partial r} + \sigma_{22}^{II}\left(\frac{u^{II}}{r} + \frac{1}{r}\frac{\partial v^{II}}{\partial \theta}\right) + \right. \\
&\qquad\qquad \left. \sigma_{12}^{II}\left(\frac{\partial v^{II}}{dr} - \frac{v^{II}}{r} + \frac{1}{r}\frac{\partial u^{II}}{\partial \theta}\right)\right]
\end{aligned}\right\} \tag{217}$$

Introducing new constants by means of

$$\left.\begin{aligned}
C_o^I &= \frac{\partial w}{\partial z}c_o^I \;, & A_n^I &= \frac{\partial w}{\partial z}a_n^I \;, & C_n^I &= \frac{\partial w}{\partial z}c_n^I \\
B_o^{II} &= \frac{\partial w}{\partial z}b_o^{II} \;, & C_o^{II} &= \frac{\partial w}{\partial z}c_o^{II} \;, & A_n^{II} &= \frac{\partial w}{\partial z}a_n^{II} \\
B_n^{II} &= \frac{\partial w}{\partial z}b_n^{II} \;, & C_n^{II} &= \frac{\partial w}{\partial z}c_n^{II} \;, & D_n^{II} &= \frac{\partial w}{\partial z}d_n^{II}
\end{aligned}\right\} \tag{218}$$

and substituting the displacements given in equations (211) and (213) into equations (214) and (215), we find the expressions for the energies with $\epsilon_{33} = \partial w/\partial z$ as a common factor. Thus,

$$\left.\begin{aligned}
W &= \left(\frac{\partial w}{\partial z}\right)^2 \left\{ \frac{E_I}{24} a^2\pi + \frac{E_{II}}{4}\left(\frac{b^2}{\sqrt{3}} - \frac{a^2\pi}{6}\right) + \right. \\
&\qquad \left. \int_0^{\pi/6} d\theta \int_0^a \overline{W}_{1_I}\, rdr + \int_0^{\pi/6} d\theta \int_a^{\frac{b}{\cos\left(\frac{\pi}{6}-\theta\right)}} \overline{W}_{1_{II}}\, rdr \right\} \\
T &= \left(\frac{\partial w}{\partial z\,\partial t}\right)^2 \int_0^{\pi/6} d\theta \int_0^a \rho_I \left[\left(u^I\right)^2 + \left(v^I\right)^2\right] rdr + \\
&\qquad \int_0^{\pi/6} d\theta \int_a^{\frac{b}{\cos\left(\frac{\pi}{6}-\theta\right)}} \rho_{II} \left[\left(u^{II}\right)^2 + \left(v^{II}\right)^2\right] rdr + \\
&\qquad \left(\frac{\partial w}{\partial t}\right)^2 \left[\rho_I \frac{a^2\pi}{12} + \frac{\rho_{II}}{2}\left(\frac{b^2}{\sqrt{3}} - \frac{a^2\pi}{6}\right)\right]
\end{aligned}\right\} \qquad (219)$$

The constants Ω_1 , Ω_2 , Ω_3 , first discussed in Part 1, now appear in the following form, as a result of equation (219):

$$\Omega_1 = \int_0^{\pi/6} d\theta \int_0^a \rho_I \left[\left(u_z^I\right)^2 + \left(u_r^I\right)^2\right] rdr +$$

$$\int_0^{\pi/6} d\theta \int_a^{\frac{b}{\cos\left(\frac{\pi}{6} - \theta\right)}} \rho_{II} \left[\left(u^{II}\right)^2 + \left(v^{II}\right)^2\right] rdr \qquad (220)$$

$$\Omega_2 = \rho_I \frac{a^2\pi}{24} + \frac{\rho_{II}}{4}\left(\frac{b^2}{\sqrt{3}} - \frac{a^2\pi}{6}\right) \qquad (221)$$

$$\Omega_3 = \frac{E_I}{24} a^2\pi + \frac{E_{II}}{4}\left(\frac{b^2}{\sqrt{3}} - \frac{a^2\pi}{6}\right) +$$

$$\int_0^{\pi/6} d\theta \int_0^a \overline{W}_{1_I} \, rdr + \int_0^{\pi/6} d\theta \int_a^{\frac{b}{\cos\left(\frac{\pi}{6} - \theta\right)}} \overline{W}_{1_{II}} \, rdr \qquad (222)$$

PART III

NUMERICAL RESULTS

TECHNICAL DISCUSSION

This portion of the report presents the numberical values obtained when both the exact theory developed in Part I and the approximate theory of Part II are used.

Figure 14 is a plot of the nondimensional values of c/c_I against a/λ, wherein c is the phase velocity in the composite, c_I is the velocity of propagation in the fiber, a is the radius of the fiber, and λ is the wavelength.

The composites used for this comparison have the following characteristics:

$E_I = 10 \cdot 10^6$ psi	$\rho_I = 2.427 \cdot 10^{-4}$ lb-sec/in.^{4}
$E_{II} = 3.8 \cdot 10^5$ psi	$\rho_{II} = 1.159 \cdot 10^{-4}$ lb-sec/in.^{4}
$\nu_I = 0.2$	$a = 2.5 \cdot 10^{-3}$ in.
$\nu_{II} = 0.35$	$V_F = 0.65$

The values of c corresponding to the exact theory are found by solving the transcendental equation which results when the 6×6 determinant is made equal to zero, as described in Part I. Appendix VII describes the computer program used to find these roots, with Ω_1, Ω_2, Ω_3 given by formulas (87) and (89). The assumption of symmetry of revolution for the basic element is then used in both cases.

The curves in Figure 14 illustrate that the error in the velocity given by the approximate theory is less than 3 percent for a/λ smaller than 0.10. If the radius of the fiber, for example, is $2.5 \cdot 10^{-3}$ in., a wave length $\lambda = 2.5 \cdot 10^{-2}$ in. will correspond to $a/\lambda = 0.10$. The wave velocity in the fiber is

$$c_I = \sqrt{\frac{E_I}{\rho_I}} = \sqrt{\frac{10 \cdot 10^6}{2.427 \cdot 10^{-4}}} = 2.03 \cdot 10^5 \text{ in./sec}$$

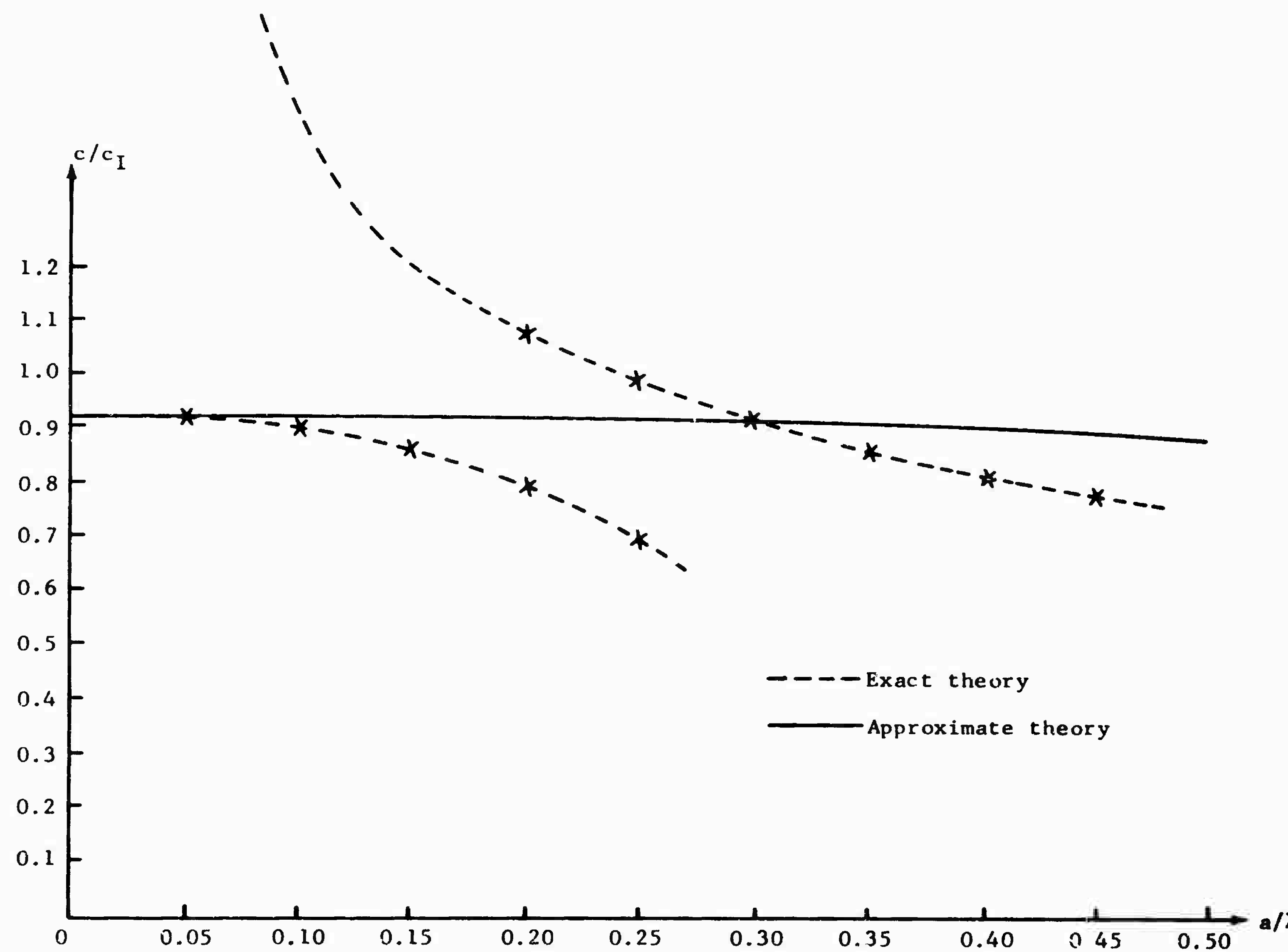

Figure 14. Phase Velocity as Determined by the Exact and Approximate Theories

Then for $a/\lambda = 0.1$, the curve corresponding to the exact theory yields the following equation:

$$c = 0.903 \cdot c_I = 1.83 \cdot 10^5 \text{ in./sec}$$

With $\lambda = 2.5 \cdot 10^{-2}$ in., the following frequency results:

$$f = \frac{c}{\lambda} = \frac{1.83 \cdot 10^5}{2.5 \cdot 10^{-2}} = 7.33 \cdot 10^6 \ \frac{1}{\text{sec}}$$

Then for any frequency smaller than $7.33 \cdot 10^6$ 1/sec, the approximate theory gives a velocity with an error of less than 3 percent.

The exact theory yields two different velocities of propagation. The implication is that two different types of waves exist, the interface waves (Raleigh waves) and those which contain combinations of dilatational and distortional waves.

A representation, where the validity of the approximate theory is more evident, is given in Figure 15, where the wave velocity in a composite c is plotted as function of the wave length in terms of fiber diameters.

The upper curve in Figure 14 corresponds to a mode which propagates only above the frequencies of 15×10^6 cycles per second. This phenomenon is clearly exhibited in Figure 16, where the wave velocity is plotted as function of the existing frequency. The frequency of 15×10^6 cycles per second is the cutoff frequency for this mode, below which this mode cannot propagate.

Appendix VIII presents the velocities for several composites. Using this parametric study, it is possible to evaluate the influence that Poisson's coefficient of the matrix and the volumetric content of fibers have on the velocity. For this computation, a ratio $a/\lambda = 0.05$ is assumed, but the values of the velocity do not change in the figures given here for any a/λ less than 0.05.

The stresses and displacements obtained in a composite of finite length, free at one end and loaded with a harmonic load at the other, will now be compared, using the exact theory as described in Part I and in formula (122) derived from the approximate theory. A composite with the following characteristics is assumed for this comparison:

$$a = 0.2500 \cdot 10^{-2} \qquad b = 0.3101 \cdot 10^{-2} \qquad V_F = 0.6000$$

$$\rho^I = 0.2428 \cdot 10^{-3} \qquad \rho^{II} = 0.1159 \cdot 10^{-3} \qquad E^I = 0.1000 \cdot [illegible]$$

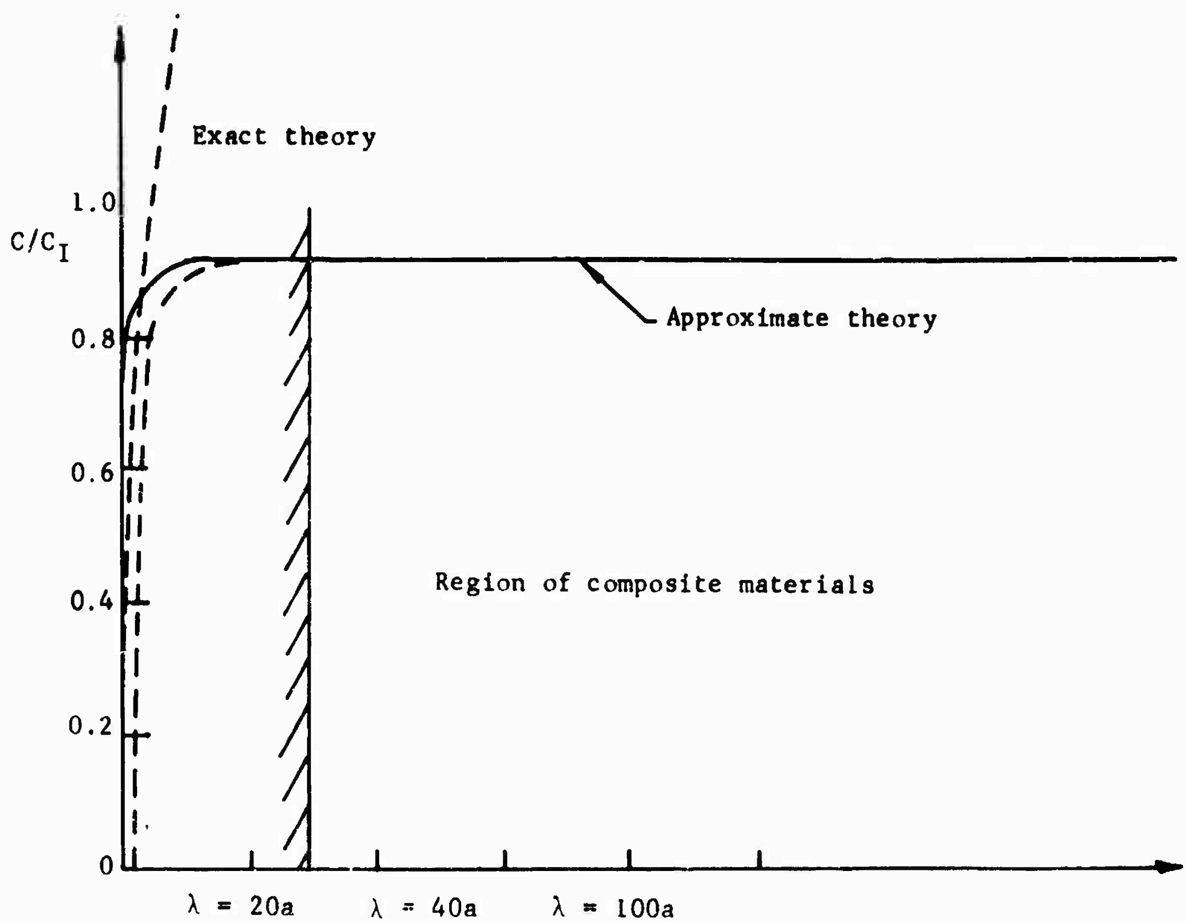

Figure 15. Wave Velocity VS. Wave Length in a Composite

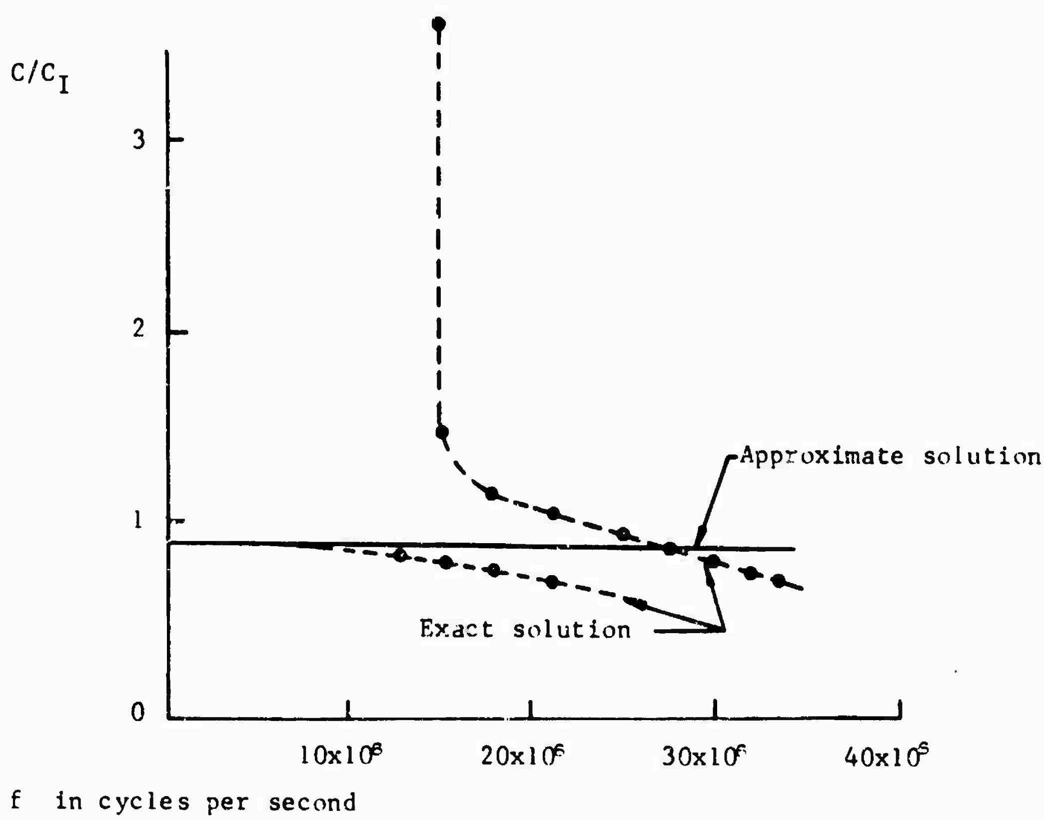

Figure 16. Propagation Velocity as Function of the Exciting Frequency as Calculated by the Exact and Approximate Theory

$$E^{II} = 0.3800 \cdot 10^6 \qquad \nu^I = 0.2000 \qquad \nu^{II} = 0.3500$$

$$L = 3.0000 \qquad \omega_e = 1.5 \cdot 10^5$$

The units are given in pounds, inches, and seconds. Table II contains the exact numbers printed by the computer. These results correspond to the cross-section $z = 1.0$. The lowest values of β are 0.80073511 (exact theory) and 0.80073 (approximate theory.

TABLE II

COMPARISON OF COMPUTER VALUES, USING EXACT AND APPROXIMATE THEORY

r	Stresses and Displacements	Exact Theory	Approximate Theory
0.00125	σ_{11}	7.54189E 05	7.542E 05
	σ_{22}	7.54189E 05	7.542E 05
	σ_{33}	1.09551E 07	1.096E 07
	u	-1.98458E -04	-1.98459E -04
	w	-1.29026E 00	-1.2903E 00
0.00250 (Fiber)	σ_{11}	7.54189E 05	7.542E 05
	σ_{22}	7.54189E 05	7.542E 05
	σ_{33}	1.095551E 07	1.096E 07
	u	-3.96916E -04	-3.96918E -04
	w	-1.29026E 00	-1.2903E 00
0.00250 (Resin)	σ_{11}	7.54183E 05	7.542E 05
	σ_{22}	4.98952E 05	4.988E 05
	σ_{33}	8.43770E 05	8.434E 05
	u	-3.96916E -04	-3.96918E -04
	w	-1.29135E 00	-1.2903E 00
0.00265	σ_{11}	7.40118E 05	7.401E 05
	σ_{22}	5.13016E 05	5.129E 05
	σ_{33}	8.43766E 05	8.434E 05
	u	-2.88343E -04	-2.88[illegible]E -04
	w	-1.29134E 00	[illegible]
0.00280	σ_{11}	7.28259E 05	7.283E 05
	σ_{22}	5.24874E 05	[illegible]
	σ_{33}	8.43762E 05	[illegible]
	u	-1.86693E -04	-1.86672E -04
	w	-1.29133E 00	-1.2903E 00
0.00295	σ_{11}	7.18159E 05	7.182E 05
	σ_{22}	5.34973E 05	5.348E 05
	σ_{33}	8.43758E 05	8.434E 05
	u	-9.08330E -05	-9.0834E -05
	w	-1.29132E 00	-1.2903E 00
0.00310	σ_{11}	7.09494E 05	7.182E 05
	σ_{22}	5.43637E 05	5.348E 05
	σ_{33}	8.43754E 05	8.434E 05
	u	-9.64452E -10	-6.36646E -12
	w	-1.29130E 00	-1.2903E 00

Formulas (220) through (222) are used to compute the constants of the differential equation of the approximate theory; a hexagonal fiber arrangement is assumed. The results presented below were obtained for a composite having the following characteristics:

$a = 0.2500 \cdot 10^{-2}$	$\nu^{I} = 0.2000$
$E^{I} = 1.0000 \cdot 10^{7}$	$\nu^{II} = 0.3000$
$E^{II} = 2.0000 \cdot 10^{5}$	$\rho^{I} = 2.4275 \cdot 10^{-4}$
	$\rho^{II} = 1.6180 \cdot 10^{-4}$

Table III presents the results for three composites having a fiber volumetric content of 0.60, 0.70, and 0.80, respectively.

TABLE III

CONSTANTS OF THE DIFFERENTIAL EQUATION FOR THE HEXAGONAL ARRANGEMENT

V_F	0.60	0.70	0.80
Ω_1	6.07319E -16	5.33778E -16	4.37757E -16
Ω_2	3.44217E -09	3.06396E -09	2.78031E -09
Ω_3	9.94555E 01	9.90068E 01	9.88233E 01

The phase velocity obtained with these values, when used in formula (112), varies less than 3 percent from the velocity found with the constants corresponding to the symmetry of revolution. However, the stress distribution in the normal plane is significantly different, especially when a high percentage of fiber is used.

CONCLUSIONS

It can be concluded that the accuracy of the approximate theory is very high. While the comparisons were performed for steady-state vibrations, the study of Figures 14 and 15 shows that the transient behavior of the composite also can be performed with the approximate theory. In fact, the predominant influence in the transient solutions is wrought by terms of low frequency; for these low frequencies, the results of the approximate theory are very accurate for the actual composites, in which a is very small.

From this analysis it is evident that existing dimensions in composite materials are not adversely affected, and the transverse shear correction can be disregarded with no appreciable affect on the accuracy of the numerical results. The transverse shear correction has been taken into account in the Mindlin-Herman theory (Ref. 16) for longitudinal vibrations of an elastic bar; the differential equations that come from this theory are completely hyperbolic, and can be solved by the numerical method of characteristics. However, in the present theory, it was possible to find closed analytical solutions even in the transient cases, which can be applied to hexagonal or other geometrical arrangements.

Appendix VI contains the computer program for determining eigenfrequencies and wavelength in a composite element; Apendix IX contains the computer program for determining Ω_1 , Ω_2 , and Ω_3 in a hexagonal multifiber element.

APPENDIX I

GENERAL SOLUTIONS OF STRESSES AND DISPLACEMENTS FOR INFINITE AND FINITE LENGTH COMPOSITES

RADIAL DISPLACEMENT

$$u = - \sum_{\alpha_1, \alpha_2 \geq 0}^{\infty} \left(\left[\sum_{\beta_1, \beta_2 \geq 0}^{\infty} \left\{ k \left[A_{1\alpha\beta} \sin(\beta_1 z) \sin(\alpha_1 c_1 t) + \right. \right. \right. \right.$$

$$A_{3\alpha\beta} \sin(\beta_1 z) \cos(\alpha_1 c_1 t) + A_{5\alpha\beta} \cos(\beta_1 z) \sin(\alpha_1 c_1 t) +$$

$$\left. A_{7\alpha\beta} \cos(\beta_1 z) \cos(\alpha_1 c_1 t) \right] \bar{\mu}_{1\alpha\beta} Z_1\left(\bar{\mu}_{1\alpha\beta} r \right) +$$

$$\left[A_{2\alpha\beta} \sin(\beta_1 z) \sin(\alpha_1 c_1 t) + A_{4\alpha\beta} \sin(\beta_1 z) \cos(\alpha_1 c_1 t) + \right.$$

$$\left. A_{6\alpha\beta} \cos(\beta_1 z) \sin(\alpha_1 c_1 t) + A_{8\alpha\beta} \cos(\beta_1 z) \cos(\alpha_1 c_1 t) \right] \bar{\mu}_{1\alpha\beta} W_1\left(\bar{\mu}_{1\alpha\beta} r \right) +$$

$$\left[B_{1\alpha\beta} \cos(\beta_2 z) \sin(\alpha_2 c_2 t) + B_{3\alpha\beta} \cos(\beta_2 z) \cos(\alpha_2 c_2 t) - \right.$$

$$\left. B_{5\alpha\beta} \sin(\beta_2 z) \sin(\alpha_2 c_2 t) - B_{7\alpha\beta} \sin(\beta_2 z) \cos(\alpha_2 c_2 t) \right] \beta_2 Z_1\left(\bar{\mu}_{1\alpha\beta} r \right) +$$

$$\left[B_{2\alpha\beta} \cos(\beta_2 z) \sin(\alpha_2 c_2 t) + B_{4\alpha\beta} \cos(\beta_2 z) \cos(\alpha_2 c_2 t) - \right.$$

$$\left. \left. \left. \left. B_{6\alpha\beta} \sin(\beta_2 z) \sin(\alpha_2 c_2 t) - B_{8\alpha\beta} \sin(\beta_3 z) \cos(\alpha_2 c_2 t) \right] \beta_2 W_1\left(\bar{\mu}_{2\alpha\beta} r \right) \right\} \right] \right) -$$

$$- \sum_{\substack{\bar{\mu}_{1\alpha}, \bar{\mu}_{2\alpha} > 0 \\ (\beta_1, \beta_2 = 0)}}^{\infty} \left\{ k \left[A_{1\alpha}\ z \sin\left(\mu_{1\alpha} c_1 t \right) + A_{3\alpha}\ z \cos\left(\mu_{1\alpha} c_1 t \right) \right] \mu_{1\alpha} Z_1\left(\bar{\mu}_{1\alpha} r \right) + \right.$$

$$+ \left[A_{2\alpha}\ z \sin(\mu_{1\alpha}c_1 t) + A_{4\alpha}\ z \cos(\mu_{1\alpha}c_1 t)\right] \mu_{1\alpha} W_1 (\bar{\mu}_{1\alpha} r) +$$

$$\left[B_{1\alpha} \sin(\mu_{2\alpha}c_2 t) + B_{3\alpha} \cos(\mu_{2\alpha}c_2 t)\right] Z_1 (\bar{\mu}_{2\alpha} r) +$$

$$\left[B_{2\alpha} \sin(\mu_{2\alpha}c_2 t) + B_{4\alpha} \cos(\mu_{2\alpha}c_2 t)\right] W_1 (\bar{\mu}_{2\alpha} r) +$$

$$A_{20} \frac{z}{r} - B_{10}\ r + (A_{60} - B_{20})\ r^{-1} \qquad (223)$$

<u>AXIAL DISPLACEMENT</u>

$$w = \sum_{\alpha_1, \alpha_2 \geq 0}^{\infty} \left(\sum_{\beta_1, \beta_2 \geq 0}^{\infty} \left\{ \left[A_{1\alpha\beta} \cos(\beta_1 z) \sin(\alpha_1 c_1 t) + \right.\right.\right.$$

$$A_{3\alpha\beta} \cos(\beta_1 z) \cos(\alpha_1 c_1 t) - A_{5\alpha\beta} \sin(\beta_1 z) \sin(\alpha_1 c_1 t) -$$

$$\left. A_{7\alpha\beta} \sin(\beta_1 z) \cos(\alpha_1 c_1 t) \right] \cdot \bar{\beta}_1 Z_0 (\bar{\mu}_{1\alpha\beta} r) +$$

$$\left[A_{2\alpha\beta} \cos(\beta_1 z) \sin(\alpha_1 c_1 t) + A_{4\alpha\beta} \cos(\beta_1 z) \cos(\alpha_1 c_1 t) - \right.$$

$$\left. A_{6\alpha\beta} \sin(\beta_1 z) \sin(\alpha_1 c_1 t) - A_{8\alpha\beta} \sin(\beta_1 z) \cos(\alpha_1 c_1 t) \right] \cdot$$

$$\beta_1 W_0 (\bar{\mu}_{1\alpha\beta} r) + \left[B_{1\alpha\beta} \sin(\beta_2 z) \sin(\alpha_2 c_2 t) + \right.$$

$$B_{3\alpha\beta} \sin(\beta_2 z) \cos(\alpha_2 c_2 t) + B_{5\alpha\beta} \cos(\beta_2 z) \sin(\alpha_2 c_2 t) +$$

$$\left. B_{7\alpha\beta} \cos(\beta_2 z) \cos(\alpha_2 c_2 t) \right] \cdot \bar{\mu}_{2\alpha\beta} Z_0 (\bar{\mu}_{2\alpha\beta} r) +$$

$$k \left[B_{2\alpha\beta} \sin(\beta_2 z) \sin(\alpha_2 c_2 t) + B_{4\alpha\beta} \sin(\beta_2 z) \cos(\alpha_2 c_2 t) + \right.$$

$$+ \; B_{8\alpha\beta} \cos(\beta_2 z) \; \sin(\alpha_2 c_2 t) + B_{8\alpha\beta} \cos(\beta_2 z) \cos(\alpha_2 c_2 t) \Big]$$

$$\bar{\mu}_{2\alpha\beta} W_0\left(\bar{\mu}_{2\alpha\beta} r\right) \Big\} \Big]\Big] + \sum_{\substack{\bar{\mu}_{1\alpha}, \bar{\mu}_{2\alpha} > o \\ (\beta_1, \beta_2 = o)}}^{\infty} \Big\{\Big[A_{1\alpha} \sin(\mu_{1\alpha} c_1 t) +$$

$$A_{3\alpha} \cos(\mu_{1\alpha} c_1 t) \Big] \; Z_0\left(\bar{\mu}_{1\alpha} r\right) + A_{2\alpha} \sin(\mu_{1\alpha} c_1 t) +$$

$$A_{4\alpha} \cos(\mu_{1\alpha} c_1 t) \Big] \; W_0\left(\bar{\mu}_{1\alpha} r\right) + \Big[B_{1\alpha} \sin(\mu_{2\alpha} c_2 t) +$$

$$B_{3\alpha} \cos(\mu_{2\alpha} c_2 t) \Big] \left(\bar{\mu}_{2\alpha} z\right) \; Z_0\left(\bar{\mu}_{2\alpha} r\right) + k \Big[B_{2\alpha} \sin(\mu_{2\alpha} c_2 t) +$$

$$B_{4\alpha} \cos(\mu_{2\alpha} c_2 t) \Big] \left(\bar{\mu}_{2\alpha} z\right) \; W_0\left(\bar{\mu}_{2\alpha} r\right) \Big\} +$$

$$A_{1o} + A_{2o} \log r + 2B_{1o} z + 2B_{5o} \qquad (224)$$

<u>NORMAL RADIAL STRESS</u>

$$\sigma_{11} = - \sum_{\alpha_1, \alpha_2 \geq o}^{\infty} \Bigg\{ \sum_{\beta_1, \beta_2' \geq o}^{\infty} \Big[\Big[\Big[A_{1\alpha\beta} \sin(\beta_1 z) \; \sin(\alpha_1 c_1 t) +$$

$$A_{3\alpha\beta} \sin(\beta_1 z) \cos(\alpha_1 c_1 t) + A_{5\alpha\beta} \cos(\beta_1 z) \; \sin(\alpha_1 c_1 t) +$$

$$A_{7\alpha\beta} \cos(\beta_1 z) \; \cos(\alpha_1 c_1 t) \Big] \cdot$$

$$\left\{ \left[\lambda \beta_1^2 + k(\lambda + 2G) \bar{\mu}_{1\alpha\beta}^2 \right] Z_0\left(\bar{\mu}_{1\alpha\beta} r\right) - \frac{2G\bar{\mu}_{1\alpha\beta} \; k Z_1\left(\bar{\mu}_{1\alpha\beta} r\right)}{r} \right\} +$$

$$\Big[A_{2\alpha\beta} \sin(\beta_1 z) \; \sin(\alpha_1 c_1 t) + A_{4\alpha\beta} \sin(\beta_1 z) \; \cos(\alpha_1 c_1 t) +$$

$$+ A_{6\alpha\beta} \cos(\beta_1 z) \sin(\alpha_1 c_1 t) + A_{8\alpha\beta} \cos(\beta_1 z) \cos(\alpha_1 c_1 t)\Big] \cdot$$

$$\left\{ \left[\lambda\beta_1^2 + k(\lambda + 2G)\bar{\mu}_{1\alpha\beta}^2\right] W_0\left(\bar{\mu}_{1\alpha\beta} r\right) - \frac{2G\bar{\mu}_{1\alpha\beta} W_1\left(\bar{\mu}_{1\alpha\beta} r\right)}{r} \right\} +$$

$$\Big[B_{1\alpha\beta} \cos(\beta_2 z) \sin(\alpha_2 c_2 t) + B_{3\alpha\beta} \cos(\beta_2 z) \cos(\alpha_2 c_2 t) -$$

$$B_{5\alpha\beta} \sin(\beta_2 z) \sin(\alpha_2 c_2 t) - B_{7\alpha\beta} \sin(\beta_2 z) \cos(\alpha_2 c_2 t)\Big] \cdot$$

$$\left(2G\bar{\mu}_{2\alpha\beta}\beta_2\right) \left[Z_0\left(\bar{\mu}_{2\alpha\beta} r\right) - \frac{Z_1\left(\bar{\mu}_{2\alpha\beta} r\right)}{\bar{\mu}_{2\alpha\beta} r}\right] +$$

$$\Big[B_{2\alpha\beta} \cos(\beta_2 z) \sin(\alpha_2 c_2 t) + B_{4\alpha\beta} \cos(\beta_2 z) \cos(\alpha_2 c_2 t) -$$

$$B_{6\alpha\beta} \sin(\beta_2 z) \sin(\alpha_2 c_2 t) - B_{8\alpha\beta} \sin(\beta_2 z) \cos(\alpha_2 c_2 t)\Big] \cdot$$

$$\left(2G\bar{\mu}_{2\alpha\beta}\beta_2\right) \left[kW_0\left(\bar{\mu}_{2\alpha\beta} r\right) - \frac{W_1\left(\bar{\mu}_{2\alpha\beta} r\right)}{\bar{\mu}_{2\alpha\beta} r}\right] \Bigg) \Bigg\} -$$

$$- \sum_{\substack{\bar{\mu}_{1\alpha}, \bar{\mu}_{2\alpha} > 0 \\ (\beta_1, \beta_2 = 0)}}^{\infty} \Bigg\{ \Big[A_{1\alpha} \sin\left(\bar{\mu}_{1\alpha} c_1 t\right) + A_{3\alpha} \cos\left(\bar{\mu}_{1\alpha} c_1 t\right)\Big] \cdot$$

$$\left(\bar{\mu}_{1\alpha}^2 z\right) \cdot k\left[(\lambda + 2G) Z_0\left(\bar{\mu}_{1\alpha} r\right) - \frac{2GZ_1\left(\bar{\mu}_{1\alpha} r\right)}{\bar{\mu}_{1\alpha} r}\right] +$$

$$\Big[A_{2\alpha} \sin\left(\bar{\mu}_{1\alpha} c_1 t\right) + A_{4\alpha} \cos\left(\bar{\mu}_{1\alpha} c_1 t\right)\Big] \cdot \left(\bar{\mu}_{1\alpha}^2 z\right) \cdot$$

$$\left[k(\lambda + 2G) W_0\left(\bar{\mu}_{1\alpha} r\right) - \frac{2GW_1\left(\bar{\mu}_{1\alpha} r\right)}{\bar{\mu}_{1\alpha} r}\right] + \Big[B_{1\alpha} \sin\left(\bar{\mu}_{2\alpha} c_2 t\right) +$$

$$B_{3\alpha} \cos\left(\bar{\mu}_{3\alpha} c_2 t\right)\Big] \left(2G\bar{\mu}_{2\alpha}\right) \left[\left(Z_0\ \bar{\mu}_{2\alpha} r\right) - \frac{Z_1\left(\bar{\mu}_{2\alpha} r\right)}{\bar{\mu}_{2\alpha} r}\right] +$$

$$\Big[B_{2\alpha} \sin\left(\bar{\mu}_{2\alpha} c_2 t\right) + B_{4\alpha} \cos\left(\bar{\mu}_{2\alpha} c_2 t\right)\Big] \cdot \left(2G\bar{\mu}_{2\alpha}\right) \cdot$$

$$\cdot \left[kW_0\left(\bar{\mu}_{2\alpha} r\right) - \frac{W_1\left(\bar{\mu}_{2\alpha} r\right)}{\bar{\mu}_{2\alpha} r} \right]\Bigg\} -$$

$$2G\left[A_{20} r^{-2} z + (A_{60} - B_{20}) r^{-2} + B_{10}\right] \tag{225}$$

NORMAL CIRCUMFERENTIAL STRESS

$$\sigma_{22} = - \sum_{\alpha_1, \alpha_2 \geq o}^{\infty} \left\{ \sum_{\beta_1, \beta_2 \geq o}^{\infty} \left[\left[\left[A_{1\alpha\beta} \sin(\beta_1 z) \sin(\alpha_1 c_1 t) + \right.\right.\right.\right.$$

$$A_{3\alpha\beta} \sin(\beta_1 z) \cos(\alpha_1 c_1 t) + A_{5\alpha\beta} \cos(\beta_1 z) \sin(\alpha_1 c_1 t) +$$

$$A_{7\alpha\beta} \cos(\beta_1 z) \cos(\alpha_1 c_1 t)\Big] \cdot \left\{ \left[\lambda\left(k\bar{\mu}_{1\alpha\beta}^2 + \beta_1^2\right)\right] Z_0\left(\bar{\mu}_{1\alpha\beta} r\right) + \right.$$

$$\left.\frac{2G\bar{\mu}_{1\alpha\beta} k Z_1\left(\bar{\mu}_{1\alpha\beta} r\right)}{r}\right\} + \Big[A_{2\alpha\beta} \sin(\beta_1 z) \sin(\alpha_1 c_1 t) +$$

$$A_{4\alpha\beta} \sin(\beta_1 z) \cos(\alpha_1 c_1 t) + A_{6\alpha\beta} \cos(\beta_1 z) \sin(\alpha_1 c_1 t) +$$

$$A_{8\alpha\beta} \cos(\beta_1 z) \cos(\alpha_1 c_1 t)\Big] \cdot \left\{ \left[\lambda\left(k\bar{\mu}_{1\alpha\beta}^2 + \beta_1^2\right)\right] W_0\left(\bar{\mu}_{1\alpha\beta} r\right) + \right.$$

$$\left.\frac{2G\bar{\mu}_{1\alpha\beta} W_1\left(\bar{\mu}_{1\alpha\beta} r\right)}{r}\right\} + \Big[B_{1\alpha\beta} \cos(\beta_2 z) \sin(\alpha_2 c_2 t) +$$

$$B_{3\alpha\beta} \cos(\beta_2 z) \cos(\alpha_2 c_2 t) - B_{5\alpha\beta} \sin(\beta_2 z) \sin(\alpha_2 c_2 t) -$$

$$B_{7\alpha\beta} \sin(\beta_2 z) \cos(\alpha_2 c_2 t)\Big] \cdot \left[(2G\beta_2) \frac{Z_1\left(\bar{\mu}_{2\alpha\beta} r\right)}{r}\right] +$$

$$\Big[B_{2\alpha\beta} \cos(\beta_2 z) \sin(\alpha_2 c_2 t) + B_{4\alpha\beta} \cos(\beta_2 z) \cos(\alpha_2 c_2 t) -$$

$$B_{6\alpha\beta} \sin(\beta_2 z) \sin(\alpha_2 c_2 t) - B_{8\alpha\beta} \sin(\beta_2 z) \cos(\alpha_2 c_2 t)\Big] \cdot$$

$$\left[(2G\beta_2)\,\frac{W_1\left(\bar{\mu}_{2\alpha\beta}r\right)}{r}\right]\Bigg]\Bigg\}\Bigg\} - \sum_{\substack{\bar{\mu}_{1\alpha},\bar{\mu}_{2\alpha}>0\\(\beta_1,\beta_2=0)}}^{\infty}\Bigg\{k\Big[A_{1\alpha}\sin\left(\bar{\mu}_{1\alpha}c_1t\right) +$$

$$A_{3\alpha}\cos\left(\bar{\mu}_{1\alpha}c_1t\right)\Big]\cdot z\cdot\left[\lambda\bar{\mu}_{1\alpha}^2 Z_0\left(\bar{\mu}_{1\alpha}r\right) + \frac{2G\bar{\mu}_{1\alpha}Z_1\left(\bar{\mu}_{1\alpha}r\right)}{r}\right] +$$

$$\Big[A_{2\alpha}\sin\left(\bar{\mu}_{1\alpha}c_1t\right) + A_{4\alpha}\cos\left(\bar{\mu}_{1\alpha}c_1t\right)\Big]\cdot z\cdot\Big[k\lambda\bar{\mu}_{1\alpha}^2 W_0\left(\bar{\mu}_{1\alpha}r\right) +$$

$$\frac{2G\bar{\mu}_{1\alpha}W_1\left(\bar{\mu}_{1\alpha}r\right)}{r}\Big] + \Big[B_{1\alpha}\sin\left(\bar{\mu}_{2\alpha}c_2t\right) + B_{3\alpha}\cos\left(\bar{\mu}_{2\alpha}c_2t\right)\Big]\cdot$$

$$(2G)\cdot\frac{Z_1\left(\bar{\mu}_{2\alpha}r\right)}{r} + \Big[B_{2\alpha}\sin\left(\bar{\mu}_{2\alpha}c_2t\right) + B_{4\alpha}\cos\left(\bar{\mu}_{2\alpha}c_2t\right)\Big]\cdot$$

$$(2G)\cdot\frac{W_1\left(\bar{\mu}_{2\alpha}r\right)}{r}\Bigg\} + 2G\Big[A_{20}r^{-2}z + (A_{80} - B_{20})r^{-2} - B_{10}\Big] \qquad (226)$$

NORMAL AXIAL STRESS

$$\sigma_{33} = -\sum_{\alpha_1,\alpha_2>0}^{\infty}\Bigg\{\sum_{\beta_1,\beta_2>0}^{\infty}\Bigg\{\Bigg[\Big[A_{1\alpha\beta}\sin(\beta_1z)\sin(\alpha_1c_1t) +$$

$$A_{3\alpha\beta}\sin(\beta_1z)\cos(\alpha_1c_1t) + A_{5\alpha\beta}\cos(\beta_1z)\sin(\alpha_1c_1t) +$$

$$A_{7\alpha\beta}\cos(\beta_1z)\cos(\alpha_1c_1t)\Big]\cdot\Big\{\Big[(\lambda + 2G)\beta_1^2 + k\lambda\bar{\mu}_{1\alpha\beta}^2\Big]Z_0\left(\bar{\mu}_{1\alpha\beta}r\right)\Big\} +$$

$$\Big[A_{2\alpha\beta}\sin(\beta_1z)\sin(\alpha_1c_1t) + A_{4\alpha\beta}\sin(\beta_1z)\cos(\alpha_1c_1t) +$$

$$A_{6\alpha\beta}\cos(\beta_1z)\sin(\alpha_1c_1t) + A_{8\alpha\beta}\cos(\beta_1z)\cos(\alpha_1c_1t)\Big]\cdot$$

$$\Big\{\Big[(\lambda + 2G)\beta_1^2 + k\lambda\bar{\mu}_{1\alpha\beta}^2\Big]W_0\left(\bar{\mu}_{1\alpha\beta}r\right)\Big\} - \Big[B_{1\alpha\beta}\cos(\beta_2z)\sin(\alpha_2c_2t) +$$

$$+ \; B_{3\alpha\beta} \cos(\beta_2 z) \cos(\alpha_2 c_2 t) - B_{5\alpha\beta} \sin(\beta_2 z) \sin(\alpha_2 c_2 t) -$$

$$B_{7\alpha\beta} \sin(\beta_2 z) \cos(\alpha_2 c_2 t) \Big] \cdot \Big[\left(2G\bar{\mu}_{2\alpha\beta}\beta_2\right) Z_0\left(\bar{\mu}_{2\alpha\beta} r\right) \Big] -$$

$$k \Big[B_{2\alpha\beta} \cos(\beta_2 z) \sin(\alpha_2 c_2 t) + B_{4\alpha\beta} \cos(\beta_2 z) \cos(\alpha_2 c_2 t) -$$

$$B_{6\alpha\beta} \sin(\beta_2 z) \sin(\alpha_2 c_2 t) - B_{8\alpha\beta} \sin(\beta_2 z) \cos(\alpha_2 c_2 t) \Big] \cdot$$

$$\Big[\left(2G\bar{\mu}_{2\alpha\beta}\beta_2\right) W_0\left(\bar{\mu}_{2\alpha\beta} r\right) \Big] \Big\} \Big] - \sum_{\substack{\mu_{1\alpha}, \mu_{2\alpha} > 0 \\ (\beta_1, \beta_2 = 0)}}^{\infty} \Big\{ k \Big[A_{1\alpha} \sin(\mu_{1\alpha} c_1 t) +$$

$$A_{3\alpha} \cos(\mu_{1\alpha} c_1 t) \Big] \left(\lambda \bar{\mu}_{1\alpha}^2 z\right) \; Z_0\left(\bar{\mu}_{1\alpha} r\right) + k \Big[A_{2\alpha} \sin\left(\bar{\mu}_{1\alpha} c_1 t\right) +$$

$$A_{4\alpha} \cos\left(\bar{\mu}_{1\alpha} c_1 t\right) \Big] \cdot \left(\lambda \bar{\mu}_{1\alpha}^2 z\right) \cdot W_0\left(\bar{\mu}_{1\alpha} r\right) - \Big[B_{1\alpha} \sin\left(\bar{\mu}_{2\alpha} c_2 t\right) +$$

$$B_{3\alpha} \cos\left(\bar{\mu}_{2\alpha} c_2 t\right) \Big] \cdot \left(2G\bar{\mu}_{2\alpha}\right) \cdot Z_0\left(\bar{\mu}_{2\alpha} r\right) - k \Big[B_{2\alpha} \sin\left(\bar{\mu}_{2\alpha} c_2 t\right) +$$

$$B_{4\alpha} \cos\left(\bar{\mu}_{2\alpha} c_2 t\right) \Big] \cdot \left(2G\bar{\mu}_{2\alpha}\right) \cdot W_0\left(\bar{\mu}_{2\alpha} r\right) \Big\} + 4GB_{10} \tag{227}$$

SHEAR STRESS

$$\sigma_{13} = -G \sum_{\alpha_1, \alpha_2 \geq 0}^{\infty} \Big\{ \Big[\sum_{\beta_1, \beta_2 \geq 0}^{\infty} \Big\{ k \Big[A_{1\alpha\beta} \cos(\beta_1 z) \sin(\alpha_1 c_1 t) +$$

$$A_{3\alpha\beta} \cos(\beta_1 z) \cos(\alpha_1 c_1 t) - A_{5\alpha\beta} \sin(\beta_1 z) \sin(\alpha_1 c_1 t) -$$

$$A_{7\alpha\beta} \sin(\beta_1 z) \cos(\alpha_1 c_1 t) \Big] \Big[\left(2\bar{\mu}_{1\alpha\beta}\beta_1\right) Z_1\left(\bar{\mu}_{1\alpha\beta} r\right) \Big] +$$

$$\Big[A_{2\alpha\beta} \cos(\beta_1 z) \sin(\alpha_1 c_1 t) + A_{4\alpha\beta} \cos(\beta_1 z) \cos(\alpha_1 c_1 t) -$$

$$
- A_{5\alpha\beta} \sin(\beta_1 z) \sin(\alpha_1 c_1 t) - A_{6\alpha\beta} \sin(\beta_1 z) \cos(\alpha_1 c_1 t) \Big] \cdot
$$

$$
\Big[\left(2\bar{\mu}_{1\alpha\beta}\beta_1\right) W_1 \left(\bar{\mu}_{1\alpha\beta} r\right) \Big] + \Big[B_{1\alpha\beta} \sin(\beta_2 z) \sin(\alpha_2 c_2 t) +
$$

$$
B_{3\alpha\beta} \sin(\beta_2 z) \cos(\alpha_2 c_2 t) + B_{5\alpha\beta} \cos(\beta_2 z) \sin(\alpha_2 c_2 t) +
$$

$$
B_{7\alpha\beta} \cos(\beta_2 z) \cos(\alpha_2 c_2 t) \Big] \cdot \Big[\left(k\bar{\mu}^2_{2\alpha\beta} - \beta_2^2\right) Z_1 \left(\bar{\mu}_{2\alpha\beta} r\right) \Big] +
$$

$$
\Big[B_{2\alpha\beta} \sin(\beta_2 z) \sin(\alpha_2 c_2 t) + B_{4\alpha\beta} \sin(\beta_2 z) \cos(\alpha_2 c_2 t) +
$$

$$
B_{6\alpha\beta} \cos(\beta_2 z) \sin(\alpha_2 c_2 t) + B_{8\alpha\beta} \cos(\beta_2 z) \cos(\alpha_2 c_2 t) \Big] \cdot
$$

$$
\Big[\left(k\bar{\mu}^2_{2\alpha\beta} - \beta_2^2\right) W_1 \left(\bar{\mu}_{2\alpha\beta} r\right) \Big] \Big\} \Big) \Big] - G \sum_{\substack{\mu_{1\alpha}, \mu_{2\alpha} > 0 \\ (\beta_1, \beta_2 = 0)}}^{\infty} \Big\{ k \Big[A_{1\alpha} \sin\left(\bar{\mu}_{1\alpha} c_1 t\right) +
$$

$$
A_{3\alpha} \cos\left(\bar{\mu}_{1\alpha} c_1 t\right) \Big] \left(2\bar{\mu}_{1\alpha}\right) \cdot Z_1 \left(\bar{\mu}_{1\alpha} r\right) + \Big[A_{2\alpha} \sin\left(\bar{\mu}_{1\alpha} c_1 t\right) +
$$

$$
A_{4\alpha} \cos\left(\bar{\mu}_{1\alpha} c_1 t\right) \Big] \left(2\bar{\mu}_{1\alpha}\right) \cdot W_1 \left(\bar{\mu}_{1\alpha} r\right) + k \Big[B_{1\alpha} \sin\left(\bar{\mu}_{2\alpha} c_2 t\right) +
$$

$$
B_{3\alpha} \cos\left(\bar{\mu}_{2\alpha} c_2 t\right) \Big] \left(\bar{\mu}^2_{2\alpha} z\right) \cdot Z_1 \left(\bar{\mu}_{2\alpha} r\right) + k \Big[B_{2\alpha} \sin\left(\bar{\mu}_{2\alpha} c_2 t\right) +
$$

$$
B_{4\alpha} \cos\left(\bar{\mu}_{2\alpha} c_2 t\right) \Big] \cdot \left(\bar{\mu}^2_{2\alpha} z\right) \cdot W_1 \left(\bar{\mu}_{2\alpha} r\right) \Big\} + 2GA_{20} r^{-1} \qquad (228)
$$

APPENDIX II

GENERAL SOLUTIONS OF STRESSES AND DISPLACEMENTS FOR THE CASE OF SEMI-INFINITE LENGTH COMPOSITE

RADIAL DISPLACEMENT

$$u = - \sum_{\alpha_1, \alpha_2 \geq o}^{\infty} \left\{ \sum_{\bar{\beta}_1, \bar{\beta}_2 \geq o}^{\infty} \left\{ \left[\bar{A}_{5\alpha\beta}\, e^{-\bar{\beta}_1 z} \sin(\alpha_1 c_1 t) + \right. \right. \right.$$

$$\left. \bar{A}_{7\alpha\beta}\, e^{-\bar{\beta}_1 z} \cos(\alpha_1 c_1 t) \right] \cdot \mu_{1\alpha\beta} J_1(\mu_{1\alpha\beta} r) +$$

$$\left[\bar{A}_{6\alpha\beta}\, e^{-\bar{\beta}_1 z} \sin(\alpha_1 c_1 t) + \bar{A}_{8\alpha\beta}\, e^{-\bar{\beta}_1 z} \cos(\alpha_1 c_1 t) \right] \cdot$$

$$\mu_{1\alpha\beta} Y_1(\mu_{1\alpha\beta} r) - \left[\bar{B}_{5\alpha\beta}\, e^{-\bar{\beta}_2 z} \sin(\alpha_2 c_2 t) + \right.$$

$$\left. \bar{B}_{7\alpha\beta}\, e^{-\bar{\beta}_2 z} \cos(\alpha_2 c_2 t) \right] \cdot \bar{\beta}_2 J_1(\mu_{2\alpha\beta} r) -$$

$$\left[\bar{B}_{6\alpha\beta}\, e^{-\bar{\beta}_2 z} \sin(\alpha_2 c_2 t) + \bar{B}_{8\alpha\beta}\, e^{-\bar{\beta}_2 z} \cos(\alpha_2 c_2 t) \right] \cdot$$

$$\left. \left. \bar{\beta}_2 Y_1(\mu_{2\alpha\beta} r) \right\} \right\} + (\bar{A}_{80} - \bar{B}_{20}) r^{-1} \tag{229}$$

AXIAL DISPLACEMENT

$$w = \sum_{\alpha_1, \alpha_2 \geq o}^{\infty} \left\{ \sum_{\bar{\beta}_1, \bar{\beta}_2 \geq o}^{\infty} \left\{ - \left[\bar{A}_{5\alpha\beta}\, e^{-\bar{\beta}_1 z} \sin(\alpha_1 c_1 t) + \right. \right. \right.$$

$$\left. \bar{A}_{7\alpha\beta}\, e^{-\bar{\beta}_1 z} \cos(\alpha_1 c_1 t) \right] \times \bar{\beta}_1 J_0(\mu_{1\alpha\beta} r) -$$

$$\left[\bar{A}_{6\alpha\beta}\, e^{-\bar{\beta}_1 z} \sin(\alpha_1 c_1 t) + \bar{A}_{8\alpha\beta}\, e^{-\bar{\beta}_1 z} \cos(\alpha_1 c_1 t) \right] \cdot$$

$$\cdot\ \bar{\beta}_1\ Y_0(\mu_{1\alpha\beta} r) + \left[\bar{B}_{5\alpha\beta}\ e^{-\bar{\beta}_2 z}\ \sin(\alpha_2 c_2 t) +\right.$$

$$\left.\bar{B}_{7\alpha\beta}\ e^{-\bar{\beta}_2 z}\ \cos(\alpha_2 c_2 t)\right] \cdot\ \mu_{2\alpha\beta}\ J_0(\mu_{2\alpha\beta} r) +$$

$$\left[\bar{B}_{6\alpha\beta}\ e^{-\bar{\beta}_2 z}\ \sin(\alpha_2 c_2 t) + \bar{B}_{8\alpha\beta}\ e^{-\bar{\beta}_2 z}\ \cos(\alpha_2 c_2 t)\right]$$

$$\left.\left.\cdot\ \mu_{2\alpha\beta}\ Y_0(\mu_{2\alpha\beta} r)\right\}\right] + \bar{A}_{10} + 2\bar{B}_{50} \qquad (230)$$

NORMAL RADIAL STRESS

$$\sigma_{11} = \sum_{\alpha_1, \alpha_2 \geq o}^{\infty} \left\| \sum_{\bar{\beta}_1, \bar{\beta}_2 \geq o}^{\infty} \left(\left\{\left[\bar{A}_{5\alpha\beta}\ e^{-\bar{\beta}_1 z}\ \sin(\alpha_1 c_1 t) +\right.\right.\right.\right.$$

$$\left.\bar{A}_{7\alpha\beta}\ e^{-\bar{\beta}_1 z}\ \cos(\alpha_1 c_1 t)\right] \cdot \left\{\left[\lambda\bar{\beta}_1^2 - (\lambda + 2G)\ \mu^2_{1\alpha\beta}\right] J_0(\mu_{1\alpha\beta} r) +\right.$$

$$\left.\frac{2G\mu_{1\alpha\beta}\ J_1(\mu_{1\alpha\beta} r)}{r}\right\} + \left[\bar{A}_{6\alpha\beta}\ e^{-\bar{\beta}_1 z}\ \sin(\alpha_1 c_1 t) +\right.$$

$$\left.\bar{A}_{8\alpha\beta}\ e^{-\bar{\beta}_1 z}\ \cos(\alpha_1 c_1 t)\right] \cdot \left\{\left[\lambda\bar{\beta}_1^2 - (\lambda + 2G)\ \mu^2_{1\alpha\beta}\right] Y_0(\mu_{1\alpha\beta} r) +\right.$$

$$\left.\frac{2G\mu_{1\alpha\beta}\ Y_1(\mu_{1\alpha\beta} r)}{r}\right\} + \left[\bar{B}_{5\alpha\beta}\ e^{-\bar{\beta}_2 z}\ \sin(\alpha_2 c_2 t) +\right.$$

$$\left.\bar{B}_{7\alpha\beta}\ e^{-\bar{\beta}_2 z}\ \cos(\alpha_2 c_2 t)\right] \cdot \left(2G_{\mu_2\alpha\beta}\ \bar{\beta}_2\right) \cdot \left[J_0(\mu_{2\alpha\beta} r) -\right.$$

$$\left.\frac{J_1(\mu_{2\alpha\beta} r)}{\mu_{2\alpha\beta} r}\right] + \left[\bar{B}_{6\alpha\beta}\ e^{-\bar{\beta}_2 z}\ \sin(\alpha_2 c_2 t) +\right.$$

$$\left.\bar{B}_{8\alpha\beta}\ e^{-\bar{\beta}_2 z}\ \cos(\alpha_2 c_2 t)\right] \cdot \left(2G_{\mu_2\alpha\beta}\ \bar{\beta}_2\right) \cdot$$

$$\left.\left.\left.\left[Y_0(\mu_{2\alpha\beta} r) - \frac{Y_1(\mu_{2\alpha\beta} r)}{\mu_{2\alpha\beta} r}\right]\right\}\right)\right\| - 2G(\bar{A}_{60} - \bar{B}_{20})r^{-2} \qquad (231)$$

NORMAL CIRCUMFERENTIAL STRESS

$$\sigma_{22} = \sum_{\alpha_1,\alpha_2\geq o}^{\infty} \left\{ \sum_{\beta_1,\beta_2\geq o}^{\infty} \left\{ \left[\overline{A}_{6\alpha\beta}\, e^{-\overline{\beta}_1 z} \sin(\alpha_1 c_1 t) + \right.\right.\right.$$

$$\left.\overline{A}_{7\alpha\beta}\, e^{-\overline{\beta}_1 z} \cos(\alpha_1 c_1 t)\right] \cdot \left[\lambda\left(\overline{\beta}_1^2 - \mu_{1\alpha\beta}^2\right) J_0(\mu_{1\alpha\beta} r) - \right.$$

$$\left.\frac{2G\mu_{1\alpha\beta}\, J_1(\mu_{1\alpha\beta} r)}{r}\right] + \left[\overline{A}_{8\alpha\beta}\, e^{-\overline{\beta}_1 z} \sin(\alpha_1 c_1 t) + \right.$$

$$\left.\overline{A}_{9\alpha\beta}\, e^{-\overline{\beta}_1 z} \cos(\alpha_1 c_1 t)\right] \cdot \left[\lambda\left(\overline{\beta}_1^2 - \mu_{1\alpha\beta}^2\right) Y_0(\mu_{1\alpha\beta} r) - \right.$$

$$\left.\frac{2G\mu_{1\alpha\beta}\, Y_1(\mu_{1\alpha\beta} r)}{r}\right] + \left[\overline{B}_{6\alpha\beta}\, e^{-\overline{\beta}_2 z} \sin(\alpha_2 c_2 t) + \right.$$

$$\left.\overline{B}_{7\alpha\beta}\, e^{-\overline{\beta}_2 z} \cos(\alpha_2 c_2 t)\right] \left[(2G\overline{\beta}_2)\, \frac{J_1(\mu_{2\alpha\beta} r)}{r}\right] +$$

$$\left[\overline{B}_{8\alpha\beta}\, e^{-\overline{\beta}_2 z} \sin(\alpha_2 c_2 t) + \overline{B}_{9\alpha\beta}\, e^{-\overline{\beta}_2 z} \cos(\alpha_2 c_2 t)\right] \cdot$$

$$\left.\left.\left[(2G\overline{\beta}_2)\, \frac{Y_1(\mu_{2\alpha\beta} r)}{r}\right]\right\}\right\} + 2G(\overline{A}_{60} - \overline{B}_{20})r^{-2} \qquad (232)$$

NORMAL AXIAL STRESS

$$\sigma_{33} = \sum_{\alpha_1,\alpha_2\geq o}^{\infty} \left\{ \sum_{\beta_1,\beta_2\geq o}^{\infty} \left\{ \left\{ \left[\overline{A}_{6\alpha\beta}\, e^{-\overline{\beta}_1 z} \sin(\alpha_1 c_1 t) + \right.\right.\right.\right.$$

$$\left.\overline{A}_{7\alpha\beta}\, e^{-\overline{\beta}_1 z} \cos(\alpha_1 c_1 t)\right] \cdot \left\{\left[(\lambda + 2G)\, \overline{\beta}_1^2 - \lambda\mu_{1\alpha\beta}^2\right] J_0(\mu_{1\alpha\beta} r)\right\} +$$

$$\left[\overline{A}_{8\alpha\beta}\, e^{-\overline{\beta}_1 z} \sin(\alpha_1 c_1 t) + \overline{A}_{9\alpha\beta}\, e^{-\overline{\beta}_1 z} \cos(\alpha_1 c_1 t)\right] \cdot$$

$$\cdot \left[(\lambda + 2G)\,\bar{\beta}_1^2 - \lambda\mu_{1\alpha\beta}^2\right] Y_0\left(\mu_{1\alpha\beta} r\right) -$$

$$\left[\bar{B}_{5\alpha\beta}\, e^{-\bar{\beta}_2 z} \sin(\alpha_2 c_2 t) + \bar{B}_{7\alpha\beta}\, e^{-\bar{\beta}_2 z} \cos(\alpha_2 c_2 t)\right] \cdot$$

$$\left[(2G\mu_{2\alpha\beta}\, \bar{\beta}_2)\, J_0(\mu_{2\alpha\beta} r)\right] - \left[\bar{B}_{6\alpha\beta}\, e^{-\bar{\beta}_2 z} \sin(\alpha_2 c_2 t) +\right.$$

$$\left.\bar{B}_{8\alpha\beta}\, e^{-\bar{\beta}_2 z} \cos(\alpha_2 c_2 t)\right] \cdot \left[(2G\mu_{2\alpha\beta}\, \bar{\beta}_2)\, Y_0(\mu_{2\alpha\beta} r)\right]\Bigg\}\Bigg) \qquad (233)$$

<u>SHEAR STRESS</u>

$$\sigma_{13} = -G \sum_{\alpha_1,\alpha_2 \geq 0}^{\infty} \Bigg(\sum_{\beta_1,\beta_2 \geq 0}^{\infty} \Bigg\{ -\left[\bar{A}_{5\alpha\beta}\, e^{-\bar{\beta}_1 z} \sin(\alpha_1 c_1 t) +\right.$$

$$\left.\bar{A}_{7\alpha\beta}\, e^{-\bar{\beta}_1 z} \cos(\alpha_1 c_1 t)\right] \cdot \left[(2\mu_{1\alpha\beta}\, \bar{\beta}_1)\, J_1(\mu_{1\alpha\beta} r)\right] -$$

$$\left[\bar{A}_{6\alpha\beta}\, e^{-\bar{\beta}_1 z} \sin(\alpha_1 c_1 t) + \bar{A}_{8\alpha\beta}\, e^{-\bar{\beta}_1 z} \cos(\alpha_1 c_1 t)\right] \cdot$$

$$\left[(2\mu_{1\alpha\beta}\, \bar{\beta}_1)\, Y_1(\mu_{1\alpha\beta} r)\right] + \left[\bar{B}_{5\alpha\beta}\, e^{-\bar{\beta}_2 z} \sin(\alpha_2 c_2 t) +\right.$$

$$\left.\bar{B}_{7\alpha\beta}\, e^{-\bar{\beta}_2 z} \cos(\alpha_2 c_2 t)\right] \cdot \left[(\mu_{2\alpha\beta}^2 + \bar{\beta}_2^2)\, J_1(\mu_{2\alpha\beta} r)\right] +$$

$$\left[\bar{B}_{6\alpha\beta}\, e^{-\bar{\beta}_2 z} \sin(\alpha_2 c_2 t) + \bar{B}_{8\alpha\beta}\, e^{-\bar{\beta}_2 z} \cos(\alpha_2 c_2 t)\right] \cdot$$

$$\left[(\mu_{2\alpha\beta}^2 + \bar{\beta}_2^2)\, Y_1(\mu_{2\alpha\beta} r)\right]\Bigg\}\Bigg) \qquad (234)$$

APPENDIX III

CHARACTERISTIC EQUATION (FREQUENCY EQUATION) FOR THE CASE OF INFINITE AND FINITE LENGTH COMPOSITE

The frequency equation for the composite of infinite and finite length is as follows.

$$\left| d_{ij} \right| = 0 \tag{235}$$

where $i,j = 1 \ldots 6$, and where

$$\begin{aligned}
d_{11} &= k\bar{\mu}_{1\alpha\beta}\, Z_1\left(\bar{\mu}_{1\alpha\beta} b\right) \\
d_{12} &= \bar{\mu}_{1\alpha\beta}\, W_1\left(\bar{\mu}_{1\alpha\beta} b\right) \\
d_{13} &= \pm\beta\, Z_1\left(\bar{\mu}_{2\alpha\beta} b\right) \\
d_{14} &= \pm\beta\, W_1\left(\bar{\mu}_{2\alpha\beta} b\right) \\
d_{15} &= 0 \\
d_{16} &= 0
\end{aligned} \tag{236}$$

$$\begin{aligned}
d_{21} &= 2k\, \bar{\mu}_{1\alpha\beta}\, \beta\, Z_1\left(\bar{\mu}_{1\alpha\beta} b\right) \\
d_{22} &= 2\, \bar{\mu}_{1\alpha\beta}\, \beta\, W_1\left(\bar{\mu}_{1\alpha\beta} b\right) \\
d_{23} &= \mp\left(k\bar{\mu}^2_{2\alpha\beta} - \beta^2\right) Z_1\left(\bar{\mu}_{2\alpha\beta} b\right) \\
d_{24} &= \mp\left(k\bar{\mu}^2_{2\alpha\beta} - \beta^2\right) W_1\left(\bar{\mu}_{2\alpha\beta} b\right) \\
d_{25} &= 0 \\
d_{26} &= 0
\end{aligned} \tag{237}$$

$$d_{31} = k\bar{u}_{1\alpha\beta}\, Z_1\left(\bar{\mu}_{1\alpha\beta}a\right)$$

$$d_{32} = \bar{u}_{1\alpha\beta}\, W_1\left(\bar{u}_{1\alpha\beta}a\right)$$

$$d_{33} = \pm\beta\, Z_1\left(\bar{u}_{2\alpha\beta}a\right)$$

$$d_{34} = \pm\beta\, W_1\left(\bar{\mu}_{2\alpha\beta}a\right)$$

$$d_{35} = k\bar{u}_{1\gamma\delta}\, Z_1\left(\bar{u}_{1\gamma\delta}a\right)$$

$$d_{36} = \mp\beta\, Z_1\left(\bar{u}_{2\gamma\delta}a\right) \tag{238}$$

$$d_{41} = \beta\, Z_0\left(\bar{u}_{1\alpha\beta}a\right)$$

$$d_{42} = \beta\, W_0\left(\bar{u}_{1\alpha\beta}a\right)$$

$$d_{43} = \mp\bar{\mu}_{2\alpha\beta}\, Z_0\left(\bar{u}_{2\alpha\beta}a\right)$$

$$d_{44} = \mp k\bar{u}_{2\alpha\beta}\, W_0\left(\bar{u}_{2\alpha\beta}a\right)$$

$$d_{45} = -\beta\, Z_0\left(\bar{u}_{1\gamma\delta}a\right)$$

$$d_{46} = \pm\bar{u}_{2\gamma\delta}\, Z_0\left(\bar{\mu}_{2\gamma\delta}a\right) \tag{239}$$

$$d_{51} = \left[+\lambda^{II}\beta^2 + k\left(\lambda^{II} + 2G^{II}\right)\bar{u}^2_{1\alpha\beta}\right] Z_0\left(\bar{\mu}_{1\alpha\beta}a\right) - \frac{2G^{II}\bar{u}_{1\alpha\beta}\, Z_1\left(\bar{\mu}_{1\alpha\beta}a\right)k}{a}$$

$$d_{52} = \left[+\lambda^{II}\beta^2 + k\left(\lambda^{II} + 2G^{II}\right)\bar{\mu}^2_{1\alpha\beta}\right] W_0\left(\bar{u}_{1\alpha\beta}a\right) - \frac{2G^{II}\bar{u}_{1\alpha\beta}\, W_1\left(\bar{u}_{1\alpha\beta}a\right)}{a}$$

$$d_{53} = \pm 2G^{II}\bar{\mu}_{2\alpha\beta}\,\beta\left[Z_0\left(\bar{\mu}_{2\alpha\beta}a\right) - \frac{Z_1\left(\bar{\mu}_{2\alpha\beta}a\right)}{\bar{\mu}_{2\alpha\beta}a}\right]$$

$$d_{54} = \pm 2G^{II}\bar{\mu}_{2\alpha\beta}\,\beta\left[k\,W_0\left(\bar{\mu}_{2\alpha\beta}a\right) - \frac{W_1\left(\bar{\mu}_{2\alpha\beta}a\right)}{\bar{\mu}_{2\alpha\beta}a}\right]$$

$$d_{55} = -\left[+\lambda^{I}\beta^2 + k\left(\lambda^{I} + 2G^{I}\right)\bar{\mu}_{1\gamma\delta}a\right]Z_0\left(\bar{\mu}_{1\gamma\delta}a\right) + \frac{2G^{I}\bar{\mu}_{1\gamma\delta}\,Z_1\left(\bar{\mu}_{1\gamma\delta}a\right)k}{a}$$

$$d_{56} = \mp 2G^{I}\bar{\mu}_{2\gamma\delta}\,\beta\left[Z_0\left(\bar{\mu}_{2\gamma\delta}a\right) - \frac{Z_1\left(\bar{\mu}_{2\gamma\delta}a\right)}{\bar{\mu}_{2\gamma\delta}a}\right] \tag{240}$$

$$d_{61} = 2kG^{II}\,\bar{\mu}_{1\alpha\beta}\,\beta\,Z_1\left(\bar{\mu}_{1\alpha\beta}a\right)$$

$$d_{62} = 2G^{II}\,\bar{\mu}_{1\alpha\beta}\,\beta\,W_1\left(\bar{\mu}_{1\alpha\beta}a\right)$$

$$d_{63} = \mp G^{II}\left(k\bar{\mu}^2_{2\alpha\beta} - \beta^2\right)Z_1\left(\bar{\mu}_{2\alpha\beta}a\right)$$

$$d_{64} = \mp G^{II}\left(k\bar{\mu}^2_{2\alpha\beta} - \beta^2\right)W_1\left(\bar{\mu}_{2\alpha\beta}a\right)$$

$$d_{65} = -2kG^{I}\bar{\mu}_{1\gamma\delta}\,\beta\,Z_1\left(\bar{\mu}_{1\gamma\delta}a\right)$$

$$d_{66} = \pm G^{I}\left(k\bar{\mu}^2_{2\gamma\delta} - \beta^2\right)Z_1\left(\bar{\mu}_{2\gamma\delta}a\right) \tag{241}$$

In the foregoing equations (235) to (241), $\bar{\mu}_{1\alpha\beta}$, $\bar{\mu}_{2\alpha\beta}$, $\bar{\mu}_{1\gamma\delta}$, $\bar{\mu}_{2\gamma\delta}$ are defined in equations (63) to (66), and the upper signs are for the case of one end free, one end fixed, and the lower signs are for the case of both ends free in finite composite.

APPENDIX IV

CHARACTERISTIC EQUATION (FREQUENCY EQUATION) FOR THE CASE OF SEMI-INFINITE LENGTH COMPOSITE

The frequency equation for the composite of semi-infinite length is as below

$$\left| \bar{d}_{ij} \right| = 0 \tag{242}$$

where $i,j = 1 \ldots 6$, and where

$$\bar{d}_{11} = \mu_{1\alpha\beta} \, J_1 \left(\mu_{1\alpha\beta} b \right)$$

$$\bar{d}_{12} = \mu_{1\alpha\beta} \, Y_1 \left(\mu_{1\alpha\beta} b \right)$$

$$\bar{d}_{13} = -\bar{\beta} \, J_1 \left(\mu_{2\alpha\beta} b \right)$$

$$\bar{d}_{14} = -\bar{\beta} \, Y_1 \left(\mu_{2\alpha\beta} b \right)$$

$$\bar{d}_{15} = 0$$

$$\bar{d}_{16} = 0 \tag{243}$$

$$\bar{d}_{21} = 2\mu_{1\alpha\beta} \, \bar{\beta} \, J_1 \left(\mu_{1\alpha\beta} b \right)$$

$$\bar{d}_{22} = 2\mu_{1\alpha\beta} \, \bar{\beta} \, Y_1 \left(\mu_{1\alpha\beta} b \right)$$

$$\bar{d}_{23} = -\left(\mu_{2\alpha\beta}^2 + \bar{\beta}^2 \right) J_1 \left(\mu_{2\alpha\beta} b \right)$$

$$\bar{d}_{24} = -\left(\mu_{2\alpha\beta}^2 + \bar{\beta}^2 \right) Y_1 \left(\mu_{2\alpha\beta} b \right)$$

$$\bar{d}_{25} = 0$$

$$\bar{d}_{26} = 0 \tag{244}$$

$$\bar{d}_{31} = \mu_{1\alpha\beta} J_1 \left(\mu_{1\alpha\beta} a \right)$$

$$\bar{d}_{32} = \mu_{1\alpha\beta} Y_1 \left(\mu_{1\alpha\beta} a \right)$$

$$\bar{d}_{33} = -\bar{\beta} J_1 \left(\mu_{2\alpha\beta} a \right)$$

$$\bar{d}_{34} = -\bar{\beta} Y_1 \left(\mu_{2\alpha\beta} a \right)$$

$$\bar{d}_{35} = -\mu_{1\gamma\delta} J_1 \left(\mu_{1\gamma\delta} a \right)$$

$$\bar{d}_{36} = +\bar{\beta} J_1 \left(\mu_{2\gamma\delta} a \right) \tag{245}$$

$$\bar{d}_{41} = \bar{\beta} J_0 \left(\mu_{1\alpha\beta} a \right)$$

$$\bar{d}_{42} = \bar{\beta} Y_0 \left(\mu_{1\alpha\beta} a \right)$$

$$\bar{d}_{43} = -\mu_{2\alpha\beta} J_0 \left(\mu_{2\alpha\beta} a \right)$$

$$\bar{d}_{44} = -\mu_{2\alpha\beta} Y_0 \left(\mu_{2\alpha\beta} a \right)$$

$$\bar{d}_{45} = -\bar{\beta} J_0 \left(\mu_{1\gamma\delta} a \right)$$

$$\bar{d}_{46} = +\mu_{2\gamma\delta} Y_0 \left(\mu_{2\gamma\delta} a \right) \tag{246}$$

$$\bar{d}_{51} = \left[\lambda^{II} \bar{\beta}^2 - \left(\lambda^{II} + 2G^{II} \right) \mu_{1\alpha\beta}^2 \right] J_0 \left(\mu_{1\alpha\beta} a \right) + \frac{2G^{II} \mu_{1\alpha\beta} J_1 \left(\mu_{1\alpha\beta} a \right)}{a}$$

$$\bar{d}_{52} = \left[\lambda^{II} \bar{\beta}^2 - \left(\lambda^{II} + 2G^{II} \right) \mu_{1\alpha\beta}^2 \right] Y_0 \left(\mu_{1\alpha\beta} a \right) + \frac{2G^{II} \mu_{1\alpha\beta} Y_1 \left(\mu_{1\alpha\beta} a \right)}{a}$$

$$\bar{d}_{53} = 2G^{II}\mu_{2\alpha\beta}\bar{\beta}\left[J_0\left(\mu_{2\alpha\beta}a\right) - \frac{J_1\left(\mu_{2\alpha\beta}a\right)}{\mu_{2\alpha\beta}a}\right]$$

$$\bar{d}_{54} = 2G^{II}\mu_{2\alpha\beta}\,\bar{\beta}\left[Y_0\left(\mu_{2\alpha\beta}a\right) - \frac{Y_1\left(\mu_{2\alpha\beta}a\right)}{\mu_{2\alpha\beta}a}\right]$$

$$\bar{d}_{55} = -\left[\lambda^{I}\bar{\beta}^{2} - \left(\lambda^{I} + 2G^{I}\right)\mu_{1\gamma\delta}^{2}\right]J_0\left(\mu_{1\gamma\delta}a\right) -$$

$$\frac{2G^{I}\mu_{1\gamma\delta}\;J_1\left(\mu_{1\gamma\delta}a\right)}{a}$$

$$\bar{d}_{56} = -2G^{I}\mu_{2\gamma\delta}\;\bar{\beta}\left[J_0\left(\mu_{2\gamma\delta}a\right) - \frac{J_1\left(\mu_{2\gamma\delta}a\right)}{\mu_{2\gamma\delta}a}\right] \qquad (247)$$

$$\bar{d}_{61} = 2G^{II}\mu_{1\alpha\beta}\;\bar{\beta}\;J_1\left(\mu_{1\alpha\beta}a\right)$$

$$\bar{d}_{62} = 2G^{II}\mu_{1\alpha\beta}\;\bar{\beta}\;Y_1\left(\mu_{1\alpha\beta}a\right)$$

$$\bar{d}_{63} = -G^{II}\left(\mu_{2\alpha\beta}^{2} + \bar{\beta}^{2}\right)J_1\left(\mu_{2\alpha\beta}a\right)$$

$$\bar{d}_{64} = -G^{II}\left(\mu_{2\alpha\beta}^{2} + \bar{\beta}^{2}\right)Y_1\left(\mu_{2\alpha\beta}a\right)$$

$$\bar{d}_{65} = -2G^{I}\mu_{1\gamma\delta}\;\bar{\beta}\;J_1\left(\mu_{1\gamma\delta}a\right)$$

$$\bar{d}_{66} = +G^{I}\left(\mu_{2\gamma\delta}^{2} + \bar{\beta}^{2}\right)J_1\left(\mu_{2\gamma\delta}a\right) \qquad (248)$$

In these equations (243) to (248), $\mu_{1\alpha\beta}$, $\mu_{2\alpha\beta}$, $\mu_{1\gamma\delta}$, $\mu_{2\gamma\delta}$, are defined by (72) to (75).

Equation (242) is a transcedental equation which relates circular frequency ω_α to $\bar{\beta}$ for given physical and geometrical values of constituents.

It must be mentioned here that the imposition of boundary conditions also gives

$$A_{20} = B_{10} = 0$$

$$A_{60} = B_{20}$$

$$A_{10} = C_{10}$$

$$B_{50} = D_{50}$$

which have no effect on frequency. These results are of no interest to us.

APPENDIX V

SOLUTIONS FOR COMPOSITE OF FINITE LENGTH WITH ONE END ($z = 0$) FIXED AND THE OTHER END ($z = L$) UNDER AXIAL PIECEWISE-CONSTANT LOADING

By applying boundary conditions (53) onto equations (223) and (224), we have

$$
\begin{aligned}
& A_{1\alpha\beta} = A_{3\alpha\beta} = A_{2\alpha\beta} = A_{4\alpha\beta} = B_{5\alpha\beta} = B_{7\alpha\beta} \\
&= B_{6\alpha\beta} = B_{8\alpha\beta} = A_{1\alpha} = A_{3\alpha} = A_{2\alpha} = A_{4\alpha} \\
&= A_{20} = C_{1\gamma\delta} = C_{3\gamma\delta} = C_{2\gamma\delta} = C_{4\gamma\delta} = D_{5\gamma\delta} \\
&= D_{7\gamma\delta} = D_{6\gamma\delta} = D_{8\gamma\delta} = C_{1\gamma} = C_{3\gamma} = C_{2\gamma} \\
&= C_{4\gamma} = C_{20} = 0
\end{aligned}
\tag{249}
$$

and

$$A_{10} + 2\, B_{50} = C_{10} + 2\, D_{50} = 0 \tag{250}$$

The Fourier expansion of the piecewise-constant function 1 is

$$\frac{4}{\pi} \sum_{n=1,2,3,\ldots}^{\infty} \frac{1}{2n-1} \sin \frac{2(2n-1)\pi t}{T} \tag{251}$$

Substituting boundary conditions (51), with the introduction of equation (251) into equation (227) we obtain

$$\frac{4P}{\pi} \sum_{n=1,2,3,\ldots}^{\infty} \frac{1}{2n-1} \sin \frac{2(2n-1)\pi t}{T}$$

$$= - \sum_{\alpha>o}^{\infty} \left\| \sum_{\beta\geq o}^{\infty} \left[\!\!\left[2\pi \int_o^a r \left\{ C_{6\gamma\delta} \left[\left(\lambda^{I} + 2G^{I} \right) \beta^2 + k\lambda^{I} \bar{\mu}_{1\gamma\delta}^{2} \right] Z_0 \left(\bar{\mu}_{1\gamma\delta} r \right) - D_{1\gamma\delta} \left(2G^{I} \bar{\mu}_{2\gamma\delta} \beta \right) Z_0 \left(\bar{\mu}_{2\gamma\delta} r \right) \right\} dr + 2\pi \int_a^b r \left\{ A_{6\alpha\beta} \left[\left(\lambda^{II} + 2G^{II} \right) \beta^2 + k\lambda^{II} \bar{\mu}_{1\alpha\beta}^{2} \right] Z_0 \left(\bar{\mu}_{1\alpha\beta} r \right) + A_{8\alpha\beta} \left[\left(\lambda^{II} + 2G^{II} \right) \beta^2 + k\lambda^{II} \bar{\mu}_{1\alpha\beta}^{2} \right] W_0 \left(\bar{\mu}_{1\alpha\beta} r \right) - B_{1\alpha\beta} \left(2G^{II} \bar{\mu}_{2\alpha\beta} \beta \right) Z_0 \left(\bar{\mu}_{2\alpha\beta} r \right) - B_{2\alpha\beta} \left(2G^{II} \bar{\mu}_{2\alpha\beta} \beta \right) kW_0 \left(\bar{\mu}_{2\alpha\beta} r \right) \right\} dr \right]\!\!\right] \right\| \cos(\beta L) \quad \sin(\omega_\alpha t) \tag{252}$$

where $\bar{\mu}_{1\alpha\beta}$, $\bar{\mu}_{2\alpha\beta}$, $\bar{\mu}_{1\gamma\delta}$, $\bar{\mu}_{2\gamma\delta}$ are moduli of $\mu_{1\alpha\beta}$, $\mu_{2\alpha\beta}$, $\mu_{1\gamma\delta}$, $\mu_{2\gamma\delta}$ respectively and where

$$\begin{aligned}
\mu_{1\alpha\beta}^2 &= \left(\frac{\omega_\alpha}{c_1^{II}} \right)^2 - \beta^2 = \left[\left(\frac{c_\alpha}{c_1^{II}} \right)^2 - 1 \right] \beta^2 \\
\mu_{2\alpha\beta}^2 &= \left(\frac{\omega_\alpha}{c_2^{II}} \right)^2 - \beta^2 = \left[\left(\frac{c_\alpha}{c_2^{II}} \right)^2 - 1 \right] \beta^2 \\
\mu_{1\gamma\delta}^2 &= \left(\frac{\omega_\alpha}{c_1^{I}} \right)^2 - \beta^2 = \left[\left(\frac{c_\alpha}{c_1^{I}} \right)^2 - 1 \right] \beta^2 \\
\mu_{2\gamma\delta}^2 &= \left(\frac{\omega_\alpha}{c_2^{I}} \right) - \beta^2 = \left[\left(\frac{c_\alpha}{c_2^{I}} \right)^2 - 1 \right] \beta^2
\end{aligned} \tag{253}$$

and

$$A_{7\alpha\beta} = A_{8\alpha\beta} = B_{3\alpha\beta} = B_{4\alpha\beta} = C_{7\alpha\beta} = D_{3\alpha\beta}$$

$$= B_{1o} = D_{1o} = 0 \qquad (254)$$

From equation (265) we also have

$$\omega_\alpha = \omega_n = \frac{2(2n-1)\pi}{T}, \qquad n=1,2,3 \qquad (255)$$

Substitution of equation (53) into equation (241) yields

$$B_{1\alpha} = B_{3\alpha} = B_{2\alpha} = B_{4\alpha} = D_{1\gamma} = D_{3\gamma} = 0 \qquad (256)$$

Applying boundary conditions of displacements and stresses at interface and outer surface, equations (45), (46), (223) through (225), and (228), we get the following relationships between coefficients

$$A_{8\alpha\beta} = \frac{|N_1\, i\, j|}{|\Delta_1\, i\, j|} A_{5\alpha\beta} = M_{1\alpha\beta} A_{5\alpha\beta}$$

$$B_{1\alpha\beta} = \frac{|N_2\, i\, j|}{|\Delta_1\, i\, j|} A_{5\alpha\beta} = M_{2\alpha\beta} A_{5\alpha\beta}$$

$$B_{2\alpha\beta} = \frac{|N_3\, i\, j|}{|\Delta_1\, i\, j|} A_{5\alpha\beta} = M_{3\alpha\beta} A_{5\alpha\beta} \qquad (257)$$

$$C_{5\gamma\delta} = \frac{|N_4\, i\, j|}{|\Delta_1\, i\, j|} A_{5\alpha\beta} = M_{4\gamma\rho} A_{5\alpha\beta}$$

$$D_{1\gamma\delta} = \frac{|N_5\, i\, j|}{|\Delta_1\, i\, j|} A_{5\alpha\beta} = M_{5\alpha\beta} A_{5\alpha\beta}$$

where determinants $|N_{1\,ij}|$, $|N_{2\,ij}|$, $|N_{3\,ij}|$, $|N_{4\,ij}|$, $|N_{5\,ij}|$, and $|\Delta_{1\,ij}|$ are defined as below.

For

$$|\Delta_{1\,ij}| \quad (i,j = 1,\ldots,5) ,$$

$$(\Delta_1)_{11} = \bar{\mu}_{1\alpha\beta}\, W_1\left(\bar{\mu}_{1\alpha\beta} b\right)$$

$$(\Delta_1)_{12} = +\beta\, Z_1\left(\bar{\mu}_{2\alpha\beta} b\right)$$

$$(\Delta_1)_{13} = +\beta\, W_1\left(\bar{\mu}_{2\alpha\beta} b\right)$$

$$(\Delta_1)_{14} = 0$$

$$(\Delta_1)_{15} = 0 \tag{258}$$

$$(\Delta_1)_{21} = \bar{\mu}_{1\alpha\beta}\, W_1\left(\bar{\mu}_{1\alpha\beta} a\right)$$

$$(\Delta_1)_{22} = +\beta\, Z_1\left(\bar{\mu}_{2\alpha\beta} a\right)$$

$$(\Delta_1)_{23} = +\beta\, W_1\left(\bar{\mu}_{2\alpha\beta} a\right)$$

$$(\Delta_1)_{24} = -k\bar{\mu}_{1\gamma\delta}\, Z_1\left(\bar{\mu}_{1\gamma\delta} a\right)$$

$$(\Delta_1)_{25} = -\beta\, Z_1\left(\bar{\mu}_{2\gamma\delta} a\right) \tag{259}$$

$$(\Delta_1)_{31} = \beta\, W_0\left(\bar{\mu}_{1\alpha\beta} a\right)$$

$$(\Delta_1)_{32} = -\bar{\mu}_{2\alpha\beta}\, Z_0\left(\bar{\mu}_{2\alpha\beta} a\right)$$

$$(\Delta_1)_{33} = -\bar{\mu}_{2\alpha\beta} k\, W_0\left(\bar{\mu}_{2\alpha\beta} a\right)$$

$$(\Delta_1)_{34} = -\beta\, Z_0\left(\bar{\mu}_{1\gamma\delta} a\right)$$

$$(\Delta_1)_{35} = +\bar{\mu}_{2\gamma\delta}\, Z_0\left(\bar{\mu}_{2\gamma\delta} a\right) \tag{260}$$

$$(\Delta_1)_{41} = +\left[\lambda^{II}\beta^2 + k\left(\lambda^{II} + 2G^{II}\right)\bar{u}_{1\alpha\beta}^2\right] W_0\left(\bar{u}_{1\alpha\beta}a\right) - \frac{2G^{II}\bar{u}_{1\alpha\beta}\, W_1\left(\bar{u}_{1\alpha\beta}a\right)}{a}$$

$$(\Delta_1)_{42} = +2G^{II}\bar{u}_{2\alpha\beta}\,\beta\left[Z_0\left(\bar{u}_{2\alpha\beta}a\right) - \frac{Z_1\left(\bar{u}_{2\alpha\beta}a\right)}{\bar{u}_{2\alpha\beta}a}\right]$$

$$(\Delta_1)_{43} = +2G^{II}\bar{u}_{2\alpha\beta}\,\beta k\left[W_0\left(\bar{u}_{2\alpha\beta}a\right) - \frac{kW_1\left(\bar{u}_{2\alpha\beta}a\right)}{\bar{u}_{2\alpha\beta}a}\right]$$

$$(\Delta_1)_{44} = -\left[\lambda^{I}\beta^2 + k\left(\lambda^{I} + 2G^{I}\right)\bar{u}_{1\gamma\delta}^2\right] Z_0\left(\bar{u}_{1\gamma\delta}a\right) + \frac{2G^{I}\bar{u}_{1\gamma\delta}k\; Z_1\left(\bar{u}_{1\gamma\delta}a\right)}{a}$$

$$(\Delta_1)_{45} = -2G^{I}\bar{u}_{2\gamma\delta}\,\beta\left[Z_0\left(\bar{u}_{2\gamma\delta}a\right) - \frac{Z_1\left(\bar{u}_{2\gamma\delta}a\right)}{\bar{u}_{2\gamma\delta}a}\right] \tag{261}$$

$$(\Delta_1)_{51} = 2G^{II}\bar{u}_{1\alpha\beta}\,\beta\, W_1\left(\bar{u}_{1\alpha\beta}a\right)$$

$$(\Delta_1)_{52} = -G^{II}\left(k\bar{u}_{2\alpha\beta}^2 - \beta^2\right)Z_1\left(\bar{u}_{2\alpha\beta}a\right)$$

$$(\Delta_1)_{53} = -G^{II}\left(k\bar{u}_{2\alpha\beta}^2 - \beta^2\right)W_1\left(\bar{u}_{2\alpha\beta}a\right)$$

$$(\Delta_1)_{54} = -\left(2G^{I}\bar{u}_{1\gamma\delta}\,\beta\right)Z_1\left(\bar{u}_{1\gamma\delta}a\right)$$

$$(\Delta_1)_{55} = +G^{I}\left(k\bar{u}_{2\gamma\delta}^2 - \beta^2\right)Z_1\left(u_{2\gamma\delta}a\right) \tag{262}$$

For $|N_{1ij}|$, the elements of the second through the fifth column are the same as that of $|\Delta_{1ij}|$, and the elements of the first column are as follows:

$$(N_1)_{11} = -k\,\bar{\mu}_{1\alpha\beta}\,Z_1\left(\bar{\mu}_{1\alpha\beta}b\right)$$

$$(N_1)_{21} = -k\,\bar{\mu}_{1\alpha\beta}\,Z_1\left(\bar{\mu}_{1\alpha\beta}a\right)$$

$$(N_1)_{31} = -\beta\,Z_o\left(\bar{\mu}_{1\alpha\beta}a\right)$$

$$(N_1)_{41} = -\left[\lambda^{II}\beta^2 + k\left(\lambda^{II} + 2G^{II}\right)\bar{\mu}^2_{1\alpha\beta}\right]Z_o\left(\bar{\mu}_{1\alpha\beta}a\right) + \frac{2G^{II}\bar{\mu}_{1\alpha\beta}\,Z_1\left(\bar{\mu}_{1\alpha\beta}a\right)}{a}$$

$$(N_1)_{51} = -2kG^{II}\,\bar{\mu}_{1\alpha\beta}\,\beta\,Z_1\left(\bar{\mu}_{1\alpha\beta}a\right) \tag{263}$$

For $|N_{2ij}|$, $|N_{3ij}|$, $|N_{4ij}|$, $|N_{5ij}|$, the elements of the second, third, fourth, and fifth columns are, respectively, the same as equations (263), and the rest of their elements are the same as the corresponding elements in $|\Delta_{1ij}|$.

With $A_{5\alpha\beta}$, $B_{1\alpha\beta}$, $B_{2\alpha\beta}$, $C_{5\gamma\delta}$, $D_{1\gamma\delta}$, defined in equations (257), we can rewrite equation (252) in the following manner:

$$\frac{4P}{\pi}\left(\frac{1}{2n-1}\right) = -\sum_{\beta\geq o}^{\infty} A_{5\alpha\beta}\left[\left(2\pi\int_a^b\left\{\left[\left(\lambda^{II} + 2G^{II}\right)\beta^2 + k\lambda^{II}\bar{\mu}^2_{1\alpha\beta}\right]\cdot\right.\right.\right.$$

$$Z_o\left(\bar{\mu}_{1\alpha\beta}r\right) + M_{1\alpha\beta}\left[\left(\lambda^{II} + 2G^{II}\right)\beta^2 + k\lambda^{II}\,\bar{\mu}^2_{1\alpha\beta}\right]\cdot$$

$$W_o\left(\bar{\mu}_{1\alpha\beta}r\right) - M_{2\alpha\beta}\left(2G^{II}\bar{\mu}_{2\alpha\beta}\beta\right)\cdot$$

$$\left.Z_o\left(\bar{\mu}_{2\alpha\beta}r\right) - M_{3\alpha\beta}\left(2G^{II}\bar{\mu}_{2\alpha\beta}\beta\right)k\;W_o\left(\bar{\mu}_{2\alpha\beta}r\right)\right\}rdr +$$

$$2\pi\int_o^a\left\{M_{4\gamma\beta}\left[\left(\lambda^{I} + 2G^{I}\right)\beta^2 + k\lambda^{I}\,\bar{\mu}^2_{1\gamma\delta}\right]\cdot\right.$$

$$\cdot\; Z_o\left(\bar{\mu}_{1\gamma\delta} r\right) - M_{5\alpha\beta}\left(2G^{I}\bar{\mu}_{2\gamma\delta}\beta\right) Z_o\left(\bar{\mu}_{2\gamma\delta} r\right)\Big\} \, rdr \Big]\Big] \cos(\beta L) \tag{264}$$

In this equation, β , $\bar{\mu}_{1\alpha\beta}$, $\bar{\mu}_{2\alpha\beta}$, $\bar{\mu}_{1\gamma\delta}$, $\bar{\mu}_{2\gamma\delta}$ are determined by equations (253) and the determinant $|d_{ij}|$ (Appendix III) with the use of

$$\omega_n = \frac{(2n-1)\pi}{T} \tag{265}$$

where $n = 1,2,3\ldots$.

By the concept of quasi-orthagonality, the coefficients $A_{5\alpha\beta}$ in equation (264) are represented as follows:

$$A_{5\alpha\beta}\cos(\beta L) = -\frac{4P}{\pi}\Bigg\{\Bigg[\Bigg(\chi_1^2\left(\frac{1}{2n-1}\right)2\pi\int_o^a\Bigg[\int_o^a\Bigg\{M_{4\alpha\beta}\left[\left(\lambda^I + 2G^I\right)\beta^2 + k\lambda^I\bar{\mu}^2_{1\gamma\delta}\right]\cdot$$

$$Z_o\left(\bar{\mu}_{1\gamma\delta} r\right) - M_{5\alpha\beta}\left(2G^I\bar{\mu}_{2\gamma\delta}\beta\right) Z_o\left(\bar{\mu}_{2\gamma\delta} r\right)\Bigg\} \; r'dr'\Bigg] rdr\Bigg) +$$

$$\Bigg(\chi_2^2\left(\frac{1}{2n-1}\right)2\pi\int_a^b\Bigg[\int_a^b\Bigg\{\left[\left(\lambda^{II} + 2G^{II}\right)\beta^2 + k\lambda^{II}\,\bar{\mu}^2_{1\alpha\beta}\right]\cdot$$

$$Z_o\left(\bar{\mu}_{1\alpha\beta} r\right) + M_{1\alpha\beta}\left[\left(\lambda^{II} + 2G^{II}\right)\beta^2 + k\lambda^{II}\,\bar{\mu}^2_{1\alpha\beta}\right]\cdot$$

$$W_o\left(\bar{\mu}_{1\alpha\beta} r\right) - M_{2\alpha\beta}\left(2G^{II}\,\bar{\mu}_{2\alpha\beta}\beta\right)\cdot$$

$$Z_o\left(\bar{\mu}_{2\alpha\beta} r\right) - M_{3\alpha\beta}\left(2G^{II}\,\bar{\mu}_{2\alpha\beta}\beta\right) k\;\; W_o\left(\bar{\mu}_{2\alpha\beta} r\right)\Bigg\} \; r'dr'\Bigg] rdr\Bigg)\Bigg]\Bigg\} \div$$

$$\Bigg[\Bigg(\chi_1^2\int_o^a\Bigg[2\pi\int_o^a r'\Bigg\{M_{4\alpha\beta}\left[\left(\lambda^I + 2G^I\right)\beta^2 + k\lambda^I\,\bar{\mu}^2_{1\gamma\delta}\right]\cdot$$

$$Z_o\left(\bar{\mu}_{1\gamma\delta} r\right) - M_{5\alpha\beta}\left(2G^I\,\bar{\mu}_{2\gamma\delta}\beta\right) Z_o\left(\bar{\mu}_{2\gamma\delta} r\right)\Bigg\}\; dr'\Bigg]^2 rdr +$$

$$\chi_2^2 \int_a^b \left[2\pi \int_a^b r' \left\{ \left[\left(\lambda^{II} + 2G^{II} \right) \beta^2 + k\lambda^{II} \bar{\mu}_{1\alpha\beta}^2 \right] \cdot \right.\right.$$

$$Z_o\left(\bar{\mu}_{1\alpha\beta} r\right) + M_{1\alpha\beta} \left[\left(\lambda^{II} + 2G^{II} \right) \beta^2 + k\lambda^{II} \bar{\mu}_{1\alpha\beta}^2 \right] \cdot$$

$$W_o\left(\bar{\mu}_{1\alpha\beta} r\right) - M_{2\alpha\beta} \left(2G^{II} \bar{\mu}_{2\alpha\beta} \beta \right) \cdot$$

$$\left.\left. Z_o\left(\bar{\mu}_{2\alpha\beta} r\right) - M_{3\alpha\beta} \left(2G^{II} \bar{\mu}_{2\alpha\beta} \beta \right) W_o\left(\bar{\mu}_{2\alpha\beta} r\right) \right\} dr' \right]^2 rdr \Bigg\} \qquad (266)$$

where χ_1 and χ_2 are defined as follows:

$$\chi_1^2 \int_o^a r \left[2\pi \int_a^b \left\{ M_{4\alpha\beta} \left[\left(\lambda^I + 2G^I \right) \beta^2 + \lambda^I \bar{\mu}_{1\gamma\delta}^2 \right] \cdot \right.\right.$$

$$\left. Z_o\left(\bar{\mu}_{1\gamma\delta} r\right) - M_{5\alpha\beta} \left(2G^I \bar{\mu}_{2\gamma\delta} \beta \right) Z_o\left(\bar{\mu}_{2\gamma\delta} r\right) \right\} r'dr' \cdot$$

$$2\pi \int_a^b \left\{ M_{4\underline{\alpha\beta}} \left[\left(\lambda^I + 2G^I \right) \underline{\beta}^2 + \lambda^I \bar{\mu}_{1\underline{\gamma\delta}}^2 \right] Z_o\left(\bar{\mu}_{1\underline{\gamma\delta}} r\right) - \right.$$

$$\left.\left. M_{5\underline{\alpha\beta}} \left(2G^I \bar{\mu}_{2\underline{\gamma\delta}} \underline{\beta} \right) Z_o\left(\bar{\mu}_{2\underline{\gamma\delta}} r\right) \right\} r'dr' \right] dr +$$

$$\chi_2^2 \int_a^b r \left[2\pi \int_a^b \left\{ \left[\left(\lambda^{II} + 2G^{II} \right) \beta^2 + k\lambda^{II} \bar{\mu}_{1\alpha\beta}^2 \right] \cdot \right.\right.$$

$$Z_o\left(\bar{\mu}_{1\alpha\beta} r\right) + M_{1\alpha\beta} \left[\left(\lambda^{II} + 2G^{II} \right) \beta^2 + k\lambda^{II} \bar{\mu}_{1\alpha\beta}^2 \right] \cdot$$

$$W_o\left(\bar{\mu}_{1\alpha\beta} r\right) - M_{2\alpha\beta} \left(2G^{II} \bar{\mu}_{2\alpha\beta} \beta \right) \cdot$$

$$\left. Z_o\left(\bar{\mu}_{2\alpha\beta} r\right) - M_{3\alpha\beta} \left(2G^{II} \bar{\mu}_{2\alpha\beta} \beta \right) W_o\left(\bar{\mu}_{2\alpha\beta} r\right) \right\} rdr \, .$$

$$2\pi \int_a^b \Bigg\{ \Bigg[\left(\lambda^{II} + 2G^{II} \right) \underline{\beta}^2 + k\lambda^{II} \bar{\mu}^2_{1\underline{\alpha\beta}} \Bigg] \cdot$$

$$Z_o\left(\bar{\mu}_{1\underline{\alpha\beta}} r\right) + M_{1\underline{\alpha\beta}} \Bigg[\left(\lambda^{II} + 2G^{II} \right) \underline{\beta}^2 + k\lambda^{II} \bar{\mu}^2_{1\underline{\alpha\beta}} \Bigg] \cdot$$

$$W_o\left(\bar{\mu}_{1\underline{\alpha\beta}} r\right) - M_{2\underline{\alpha\beta}} \left(2G^{II} \bar{\mu}_{2\underline{\alpha\beta}} \underline{\beta} \right) +$$

$$Z_o\left(\bar{\mu}_{2\underline{\alpha\beta}} r\right) - M_{3\underline{\alpha\beta}} \left(2G^{II} \bar{\mu}_{2\underline{\alpha\beta}} \underline{\beta} \right) \; W_o \left(\bar{\mu}_{2\underline{\alpha\beta}} r \right) \Bigg\} r' dr' \Bigg] \, dr = 0 \qquad (267)$$

With $A_{5\alpha\beta}$ found by equation (266) and the eigenvalues obtained from equations (235) through (241), we can get $A_{6\alpha\beta}$, $B_{1\alpha\beta}$, $B_{2\alpha\beta}$, $C_{5\gamma\delta}$, $D_{1\gamma\delta}$ from equations (251) through (263) and then obtain displacements and stresses of the composite from equations (223) through (228).

APPENDIX VI

SOLUTIONS FOR COMPOSITE OF FINITE LENGTH WITH ONE END (z = 0) FREELY SUPPORTED AND THE OTHER (z = L) UNDER PIECEWISE-CONSTANT LOADING

By applying boundary conditions (54) to equations (223) and (224), we have the following:

$$A_{5\alpha\beta} = A_{7\alpha\beta} = A_{6\alpha\beta} = A_{8\alpha\beta} = B_{1\alpha\beta} = B_{3\alpha\beta} =$$

$$B_{2\alpha\beta} = B_{4\alpha\beta} = B_{1\alpha} = B_{3\alpha} = B_{2\alpha} = B_{4\alpha} =$$

$$B_{10} = C_{5\gamma\delta} = C_{7\gamma\delta} = C_{6\gamma\delta} = C_{8\gamma\delta} = D_{1\gamma\delta} =$$

$$D_{3\gamma\delta} = D_{2\gamma\delta} = D_{4\gamma\delta} = D_{1\gamma} = D_{2\gamma} = D_{3\gamma} =$$

$$D_{4\gamma} = D_{10} = 0 \tag{268}$$

and

$$A_{60} - B_{20} = C_{60} - D_{20} \tag{269}$$

The Fourier expansion of the piecewise-constant function 1 is:

$$\frac{4}{\pi} \sum_{n=1,2,3...}^{\infty} \frac{1}{2n-1} \sin \frac{2(2n-1)\pi t}{T} \tag{270}$$

Substituting boundary conditions (55) into equation (227), and also introducing equation (270), we obtain

$$\frac{4P}{\pi} \sum_{n=1,2,3...}^{\infty} \frac{1}{2n-1} \sin \frac{2(2n-1)\pi t}{T} = -\sum_{\alpha>0}^{\infty} \left\{ \left[\sum_{\beta\geq 0}^{\infty} 2\pi \int_0^a r \cdot \right.\right.$$

$$\left\{ C_{1\gamma\delta} \left[\left(\lambda^I + 2G^I \right) \beta^2 + k\lambda^I \bar{u}^2_{1\gamma\delta} \right] Z_0\left(\bar{u}_{1\gamma\delta} r \right) + \right.$$

$$+ D_{5\gamma\delta} \left(2G^{I}\, \bar{\mu}_{2\gamma\delta}\beta\right) Z_o\left(\bar{\mu}_{2\gamma\delta} r\right) \Big\} \, dr \, +$$

$$2\pi \int_a^b r \Big\{ A_{1\alpha\beta} \left[\left(\lambda^{II} + 2G^{II}\right)\beta^2 + k\lambda^{II}\, \bar{\mu}^2_{1\alpha\beta}\right] \cdot$$

$$Z_o\left(\bar{\mu}_{1\alpha\beta} r\right) + A_{2\alpha\beta} \left[\left(\lambda^{II} + 2G^{II}\right)\beta^2 + k\lambda^{II}\bar{\mu}^2_{1\alpha\beta}\right] \cdot$$

$$W_o\left(\bar{\mu}_{1\alpha\beta} r\right) + B_{5\alpha\beta} \left(2G^{II}\, \bar{\mu}_{2\alpha\beta}\beta\right) \cdot$$

$$Z_o\left(\bar{\mu}_{2\alpha\beta} r\right) + B_{6\alpha\beta} \left(2G^{II}\, \bar{\mu}_{2\alpha\beta}\beta\right) k \cdot$$

$$W_o\left(\bar{\mu}_{2\alpha\beta} r\right) \Big\} dr \Bigg) \sin(\beta L) \sin(\omega_\alpha t) \qquad (271)$$

where $\bar{\mu}_{1\alpha\beta}$, $\bar{\mu}_{2\alpha\beta}$, $\bar{\mu}_{1\gamma\delta}$, $\bar{\mu}_{2\gamma\delta}$ are moduli of $\mu_{1\alpha\beta}$, $\mu_{2\alpha\beta}$, $\mu_{1\gamma\delta}$, $\mu_{2\gamma\delta}$, respectively, and where

$$\mu^2_{1\alpha\beta} = \left(\frac{\omega_\alpha}{c_1^{II}}\right)^2 - \beta^2 = \left[\left(\frac{c_\alpha}{c_1^{II}}\right)^2 - 1\right]\beta^2$$

$$\mu^2_{2\alpha\beta} = \left(\frac{\omega_\alpha}{c_2^{II}}\right)^2 - \beta^2 = \left[\left(\frac{c_\alpha}{c_2^{II}}\right)^2 - 1\right]\beta^2$$

$$\mu^2_{1\gamma\delta} = \left(\frac{\omega_\alpha}{c_1^{I}}\right)^2 - \beta^2 = \left[\left(\frac{c_\alpha}{c_1^{I}}\right)^2 - 1\right]\beta^2$$

$$\mu^2_{2\gamma\delta} = \left(\frac{\omega_\alpha}{c_2^{I}}\right)^2 - \beta^2 = \left[\left(\frac{c_\alpha}{c_2^{I}}\right)^2 - 1\right]\beta^2 \qquad (272)$$

and

$$A_{3\alpha\beta} = A_{4\alpha\beta} = B_{7\alpha\beta} = B_{8\alpha\beta} = C_{3\alpha\beta} = D_{7\alpha\beta} =$$

$$B_{10} = D_{10} = 0 \qquad (273)$$

From equation (271), we also have

$$\omega_\alpha = \omega_n = \frac{2(2n-1)\pi}{T} \tag{274}$$

where $n = 1,2,3\ldots$.

Substituting equation (57) into equation (228) yields

$$A_{1\alpha} = A_{3\alpha} = A_{2\alpha} = A_{4\alpha} = C_{1\gamma} = C_{3\gamma} = 0 \tag{275}$$

Applying the boundary conditions of the displacements and stresses at the interface and the outer surface — in other words, equations (45), (46), (223) through (225), and (228) — we obtain the following relationship between coefficients:

$$A_{2\alpha\beta} = \frac{|N_{6\,ij}|}{|\Delta_{2\,ij}|} A_{1\alpha\beta} = M_{6\alpha\beta} A_{1\alpha\beta}$$

$$B_{5\alpha\beta} = \frac{|N_{7\,ij}|}{|\Delta_{2\,ij}|} A_{1\alpha\beta} = M_{7\alpha\beta} A_{1\alpha\beta}$$

$$B_{6\alpha\beta} = \frac{|N_{8\,ij}|}{|\Delta_{2\,ij}|} A_{1\alpha\beta} = M_{8\alpha\beta} A_{1\alpha\beta}$$

$$C_{1\gamma\delta} = \frac{|N_{9\,ij}|}{|\Delta_{2\,ij}|} A_{1\alpha\beta} = M_{9\alpha\beta} A_{1\alpha\beta}$$

$$D_{5\gamma\delta} = \frac{|N_{10\,ij}|}{|\Delta_{2\,ij}|} A_{1\alpha\beta} = M_{10\alpha\beta} A_{1\alpha\beta} \tag{276}$$

where the determinants $|N_{6\,ij}|$, $|N_{7\,ij}|$, $|N_{8\,ij}|$, $|N_{9\,ij}|$, $|N_{10\,ij}|$, and $|\Delta_{2\,ij}|$ are defined as follows:

For $|\Delta_{2\,ij}|$, with $i,j = 1\ldots5$,

$$(\Delta_2)_{11} = \bar{\mu}_{1\alpha\beta} W_1\left(\bar{\mu}_{1\alpha\beta} b\right)$$

$$(\Delta_2)_{12} = -\beta Z_1\left(\bar{\mu}_{2\alpha\beta} b\right)$$

$$(\Delta_2)_{13} = -\beta W_1\left(\bar{\mu}_{2\alpha\beta} b\right)$$

$$(\Delta_2)_{14} = 0$$

$$(\Delta_2)_{15} = 0 \tag{277}$$

$$(\Delta_2)_{21} = \bar{\mu}_{1\alpha\beta} W_1\left(\bar{\mu}_{1\alpha\beta} a\right)$$

$$(\Delta_2)_{22} = -\beta Z_1\left(\bar{\mu}_{2\alpha\beta} a\right)$$

$$(\Delta_2)_{23} = -\beta W_1\left(\bar{\mu}_{2\alpha\beta} a\right)$$

$$(\Delta_2)_{24} = -k \bar{\mu}_{1\gamma\delta} Z_1\left(\bar{\mu}_{1\gamma\delta} a\right)$$

$$(\Delta_2)_{25} = +\beta Z_1\left(\bar{\mu}_{1\gamma\delta} a\right) \tag{278}$$

$$(\Delta_2)_{31} = \beta W_o\left(\bar{\mu}_{1\alpha\beta} a\right)$$

$$(\Delta_2)_{32} = +\bar{\mu}_{2\alpha\beta} Z_o\left(\bar{\mu}_{2\alpha\beta} a\right)$$

$$(\Delta_2)_{33} = +\bar{\mu}_{2\alpha\delta} k W_o\left(\bar{\mu}_{2\alpha\beta} a\right)$$

$$(\Delta_2)_{34} = -\beta Z_o\left(\bar{\mu}_{1\gamma\delta} a\right)$$

$$(\Delta_2)_{35} = -\bar{\mu}_{2\gamma\delta} Z_o\left(\bar{\mu}_{2\gamma\delta} a\right) \tag{279}$$

$$(\Delta_2)_{41} = +\left[\lambda^{II}\beta^2 + k\left(\lambda^{II} + 2G^{II}\right)\bar{\mu}^2_{1\alpha\beta}\right] W_o\left(\bar{\mu}_{1\alpha\beta} a\right) - \frac{2G^{II}\bar{\mu}_{1\alpha\beta} W_1\left(\bar{\mu}_{1\alpha\beta} a\right)}{a}$$

$$(\Delta_2)_{42} = -2G^{II}\bar{\mu}_{2\alpha\beta}\beta\left[Z_o(\bar{\mu}_{2\alpha\beta}a) - \frac{Z_1(\bar{\mu}_{2\alpha\beta}a)}{\bar{\mu}_{2\alpha\beta}a}\right]$$

$$(\Delta_2)_{43} = -2G^{II}\bar{\mu}_{2\alpha\beta}\beta k\left[W_o(\bar{\mu}_{2\alpha\beta}a) - \frac{kW_1(\bar{\mu}_{2\alpha\beta}a)}{\bar{\mu}_{2\alpha\beta}a}\right]$$

$$(\Delta_2)_{44} = -\left[\lambda^I\beta^2 + k(\lambda^I + 2G^I)\bar{\mu}^2_{1\gamma\delta}\right]Z_o(\bar{\mu}_{1\gamma\delta}a) + \frac{2G^I\bar{\mu}_{1\gamma\delta}kZ_1(\bar{\mu}_{1\gamma\delta}a)}{a}$$

$$(\Delta_2)_{45} = +2G^I\bar{\mu}_{2\gamma\delta}\beta\left[Z_o(\bar{\mu}_{2\gamma\delta}a) - \frac{Z_1(\bar{\mu}_{2\gamma\delta}a)}{\bar{\mu}_{2\gamma\delta}a}\right] \tag{280}$$

$$(\Delta_2)_{51} = 2G^{II}\bar{\mu}_{1\alpha\beta}\beta\ W_1(\bar{\mu}_{2\alpha\beta}a)$$

$$(\Delta_2)_{52} = +G^{II}(k\bar{\mu}^2_{2\alpha\beta} - \beta^2)\ Z_1(\bar{\mu}_{2\alpha\beta}a)$$

$$(\Delta_2)_{53} = +G^{II}(k\bar{u}^2_{2\alpha\beta} - \beta^2)\ W_1(\bar{u}_{2\alpha\beta}a)$$

$$(\Delta_2)_{54} = -(2G^I\bar{u}_{1\gamma\delta}\ \beta)\ Z_1(\bar{u}_{1\gamma\delta}a)\ k$$

$$(\Delta_2)_{55} = -G^I(k\bar{u}^2_{2\gamma\delta} - \beta^2)\ Z_1(\bar{u}_{2\gamma\delta}a) \tag{281}$$

For $|N_{8\,ij}|$, the elements of the second to the fifth column are the same as that of $|\Delta_{2ij}|$ and the elements of the first-column are as follows:

$$(N_8)_{11} = -k\bar{u}_{1\alpha\beta}\ Z_1(\bar{u}_{1\alpha\beta}b)$$

$$(N_8)_{21} = -k\bar{u}_{1\alpha\beta}\ Z_1(\bar{u}_{1\alpha\beta}a)$$

$$(N_8)_{31} = -\beta\ Z_o(\bar{u}_{1\alpha\beta}a)$$

$$(N_8)_{41} = -\left[\lambda^{II}\beta^2 + k(\lambda^{II} + 2G^{II})\bar{u}^2_{1\alpha\beta}\right]Z_o(\bar{u}_{1\alpha\beta}a) + \frac{2G^{II}\bar{u}_{1\alpha\beta}Z_1(\bar{u}_{1\alpha\beta}a)k}{a}$$

$$(N_8)_{51} = 2kG^{II}\bar{u}_{1\alpha\beta}\beta\ Z_1(\bar{u}_{1\alpha\beta}a) \tag{282}$$

For $|N_{7ij}|$, $|N_{8ij}|$, $|N_{9ij}|$, and $|N_{10ij}|$, the elements of the second, third, fourth, and fifth columns are, respectively, the same as those in equation (282), and the rest of their elements are the same as the corresponding elements in $|\Delta_{2ij}|$.

With $A_{2\alpha\beta}$, $B_{5\alpha\beta}$, $B_{6\alpha\beta}$, $C_{1\gamma\delta}$, $D_{5\gamma\delta}$ defined in equations (276), we can rewrite equation (271) as follows:

$$\frac{4P}{\pi}\left(\frac{1}{2n-1}\right) = -\left\| \sum_{\beta\geq o}^{\infty} A_{1\alpha\beta}\left(2\pi\int_a^b r\left\{\left[\left(\lambda^{II} + 2G^{II}\right)\beta^2 + k\lambda^{II}\,\bar{\mu}_{1\alpha\beta}^2\right]\cdot\right.\right.\right.$$

$$Z_o\left(\bar{\mu}_{1\alpha\beta}r\right) + M_{6\alpha\beta}\left[\left(\lambda^{II} + 2G^{II}\right)\beta^2 + k\lambda^{II}\,\bar{\mu}_{1\alpha\beta}^2\right]\cdot$$

$$W_o\left(\bar{\mu}_{1\alpha\beta}r\right) + M_{7\alpha\beta}\left(2G^{II}\,\bar{\mu}_{2\alpha\beta}\beta\right)\cdot$$

$$Z_o\left(\bar{\mu}_{2\alpha\beta}r\right) + M_{8\alpha\beta}\left(2G^{II}\,\bar{\mu}_{2\alpha\beta}\beta\right) k_1\, W_o\left(\bar{\mu}_{2\alpha\beta}r\right)\Big\}\,dr\, +$$

$$2\pi\int_o^a r\;\Big\{M_{9\alpha\beta}\left[\left(\lambda^{I} + 2G^{I}\right)\beta^2 + k\lambda^{I}\,\bar{\mu}_{1\gamma\delta}^2\right]\cdot$$

$$\left.\left.Z_o\left(\bar{\mu}_{1\gamma\delta}r\right) + M_{10\alpha\beta}\left(2G^{I}\,\bar{\mu}_{2\gamma\delta}\beta\right) Z_o\left(\bar{\mu}_{2\gamma\delta}r\right)\Big\}\,dr\right)\sin(\beta L)\right\| \tag{283}$$

In the equation above, β , $\bar{\mu}_{1\alpha\beta}$, $\bar{\mu}_{2\alpha\beta}$, $\bar{\mu}_{1\gamma\delta}$, $\bar{\mu}_{2\gamma\delta}$ are determined by equations (272) and the determinant $|d_{ij}|$ with the use of

$$\omega_n = \frac{2(2n-1)\pi}{T} \tag{284}$$

where $n = 1,2,3\ldots$.

By the same approach taken in Appendix V of this report, the coefficients $A_{1\alpha\beta}$ of equation (283) are represented by the equation which follows on the next page.

$$
\begin{aligned}
A_{1\alpha\beta}\sin(\beta L) = -\frac{4P}{\pi}\Bigg\{ & \chi_3^2\left(\frac{1}{2n-1}\right)2\pi\int_0^a\Bigg[\int_0^a\Big\{M_{9\alpha\beta}\Big[\left(\lambda^I + 2G^I\right)\beta^2 + k\lambda^I\,\bar{\mu}_{1\gamma\delta}^2\Big]\cdot \\
& Z_0\left(\bar{\mu}_{1\gamma\delta}r\right) + M_{10\alpha\beta}\left(2G^I\,\bar{\mu}_{2\gamma\delta}\beta\right)\,Z_0\left(\bar{\mu}_{2\gamma\delta}r\right)\Big\}\,r'dr'\Bigg]\,rdr\, + \\
& \chi_4^2\left(\frac{1}{2n-1}\right)2\pi\int_a^b\Bigg[\int_a^b\Big\{\Big[\left(\lambda^{II} + 2G^{II}\right)\beta^2 + k\lambda^{II}\,\bar{\mu}_{1\alpha\beta}^2\Big]\cdot \\
& Z_0\left(\bar{\mu}_{1\alpha\beta}r\right) + M_{6\alpha\beta}\Big[\left(\lambda^{II} + 2G^{II}\right)\beta^2 + k\lambda^{II}\,\bar{\mu}_{1\alpha\beta}^2\Big]\cdot \\
& W_0\left(\bar{\mu}_{1\alpha\beta}r\right) + M_{7\alpha\beta}\left(2G^{II}\,\bar{\mu}_{2\alpha\beta}\beta\right)\cdot \\
& Z_0\left(\bar{\mu}_{2\alpha\beta}r\right) + M_{8\alpha\beta}\left(2G^{II}\,\bar{\mu}_{2\alpha\beta}\beta\right)k\;W_0\left(\bar{\mu}_{2\alpha\beta}r\right)\Big\}\,r'dr'\Bigg]\,rdr\Bigg\} \div \\
\Bigg\{ & \chi_3^2\int_0^a\Bigg[2\pi\int_0^a r'\Big\{M_{9\alpha\beta}\Big[\left(\lambda^I + 2G^I\right)\beta^2 + k\lambda^I\,\bar{\mu}_{1\gamma\delta}^2\Big]\cdot \\
& Z_0\left(\bar{\mu}_{1\gamma\delta}r\right) + M_{10\alpha\beta}\left(2G^I\,\bar{\mu}_{2\gamma\delta}\beta\right)\,Z_0\left(\bar{\mu}_{2\gamma\delta}r\right)\Big\}\,dr'\Bigg]^2 rdr\, + \\
& \chi_4^2\int_a^b\Bigg[2\pi\int_a^b r'\Big\{\Big[\left(\lambda^{II} + 2G^{II}\right)\beta^2 + k\lambda^{II}\,\bar{\mu}_{1\alpha\beta}^2\Big]\cdot \\
& Z_0\left(\bar{\mu}_{1\alpha\beta}r\right) + M_{6\alpha\beta}\Big[\left(\lambda^{II} + 2G^{II}\right)\beta^2 + k\lambda^{II}\,\bar{\mu}_{1\alpha\beta}^2\Big]\cdot \\
& W_0\left(\bar{\mu}_{1\alpha\beta}r\right) + M_{7\alpha\beta}\left(2G^{II}\,\bar{\mu}_{2\alpha\beta}\beta\right)\cdot \\
& Z_0\left(\bar{\mu}_{2\alpha\beta}r\right) + M_{8\alpha\beta}\left(2G^{II}\,\bar{\mu}_{2\alpha\beta}\beta\right)\;W_0\left(\bar{\mu}_{2\alpha\beta}r\right)\Big\}\,dr'\Bigg]^2 rdr\Bigg\}
\end{aligned}
\qquad (285)
$$

where χ_3 and χ_4 are defined as follows:

$$\chi_3^2 \int_0^a r \left[2\pi \int_a^b \left\{ M_{9\alpha\beta} \left[\left(\lambda^I + 2G^I\right)\beta^2 + \lambda^I \bar{\mu}_{1\gamma\delta}^2 \right] Z_o\left(\bar{\mu}_{1\gamma\delta} r\right) + \right.\right.$$

$$\left. M_{10\alpha\beta} \left(2G^I \bar{\mu}_{2\gamma\delta} \beta\right) Z_o\left(\bar{\mu}_{2\gamma\delta} r\right) \right\} r'dr' \cdot$$

$$2\pi \int_a^b \left\{ M_{9\underline{\alpha\beta}} \left[\left(\lambda^I + 2G^I\right) \underline{\beta}^2 + \lambda^I \bar{\mu}_{1\underline{\gamma\delta}}^2 \right] Z_o\left(\mu_{1\underline{\gamma\delta}} r\right) + \right.$$

$$\left.\left. M_{10\underline{\alpha\beta}} \left(2G^I \bar{\mu}_{2\underline{\gamma\delta}} \underline{\beta}\right) Z_o\left(\bar{\mu}_{2\underline{\gamma\delta}} r\right) \right\} r'dr' \right] dr +$$

$$\chi_4^2 \int_a^b r \left[2\pi \int_a^b \left\{ \left[\left(\lambda^{II} + 2G^{II}\right)\beta^2 + k\lambda^{II} \bar{\mu}_{1\alpha\beta}^2 \right] Z_o\left(\bar{\mu}_{1\alpha\beta} r\right) + \right.\right.$$

$$M_{6\alpha\beta} \left[\left(\lambda^{II} + 2G^{II}\right)\beta^2 + k\lambda^{II} \bar{\mu}_{1\alpha\beta} \right] W_o\left(\bar{\mu}_{1\alpha\beta} r\right) +$$

$$M_{7\alpha\beta} \left(2G^{II} \bar{\mu}_{2\alpha\beta} \beta\right) \quad Z_o\left(\bar{\mu}_{2\alpha\beta} r\right) +$$

$$\left. M_{8\alpha\beta} \left(2G^{II} \bar{\mu}_{2\alpha\beta} \beta\right) \quad W_o\left(\bar{\mu}_{2\alpha\beta} r\right) \right\} \cdot r'dr' \cdot$$

$$2\pi \int_a^b \left\{ \left[\left(\lambda^{II} + 2G^{II}\right)\underline{\beta}^2 + k\lambda^{II} \bar{\mu}_{1\underline{\alpha\beta}}^2 \right] Z_o\left(\bar{\mu}_{1\underline{\alpha\beta}} r\right) + \right.$$

$$M_{6\underline{\alpha\beta}} \left[\left(\lambda^{II} + 2G^{II}\right)\underline{\beta}^2 + k\lambda^{II} \bar{\mu}_{1\underline{\alpha\beta}}^2 \right] W_o\left(\bar{\mu}_{1\underline{\alpha\beta}} r\right) +$$

$$M_{7\underline{\alpha\beta}} \left(2G^{II} \bar{\mu}_{2\underline{\alpha\beta}} \underline{\beta}\right) \quad Z_o\left(\bar{\mu}_{2\underline{\alpha\beta}} r\right) +$$

$$\left.\left. M_{8\underline{\alpha\beta}} \left(2G^{II} \bar{\mu}_{2\underline{\alpha\beta}} \underline{\beta}\right) W_o\left(\bar{\mu}_{2\underline{\alpha\beta}} r\right) \right\} r'dr' \right] dr = 0 \qquad (286)$$

With $A_{1\alpha\beta}$ found by equation (285) and the eigenvalues obtained from equations (235) through (241), we can get $A_{2\alpha\beta}$, $B_{5\alpha\beta}$, $B_{6\alpha\beta}$, $C_{1\gamma\delta}$, and and $D_{5\gamma\delta}$ from equations (276) and then obtain displacements and stresses of the composite from equations (223) through (228).

APPENDIX VII

SOLUTIONS FOR COMPOSITE OF SEMI-INFINITE LENGTH WITH THE END (z = 0) UNDER PIECEWISE-CONSTANT LOADING

By applying boundary conditions (58) on equations (233) and (234), we get

$$
\begin{aligned}
&\frac{4P}{\pi} \sum_{n=1,2,3\ldots}^{\infty} \frac{1}{2n-1} \sin \frac{2(2n-1)\pi t}{T} \\
&= - \sum_{n=1,2,3\ldots}^{\infty} \left\| \sum_{\beta>0}^{\infty} \left(2\pi \int_{0}^{a} r \left\{ \bar{C}_{s\gamma\delta} \left[\left(\lambda^{I} + 2G^{I}\right)\bar{\beta}^{2} - \right.\right.\right.\right. \\
&\left. \lambda^{I} \mu_{1\gamma\delta}^{2} \right] J_0(\mu_{1\gamma\delta} r) - \bar{D}_{s\gamma\delta} \left(2G^{I} \mu_{2\gamma\delta} \bar{\beta}\right) J_0(\mu_{2\gamma\delta} r) \Big\} dr \; + \\
&2\pi \int_{a}^{b} r \Big\{ \bar{A}_{s\alpha\beta} \left[\left(\lambda^{II} + 2G^{II}\right)\bar{\beta}^{2} - \lambda^{II} \mu_{1\alpha\beta}^{2} \right] J_0(\mu_{1\alpha\beta} r) \; + \\
&\bar{A}_{s\alpha\beta} \left[\left(\lambda^{II} + 2G^{II}\right)\bar{\beta}^{2} - \lambda^{II} \mu_{1\alpha\beta}^{2} \right] Y_0(\mu_{1\alpha\beta} r) \; - \\
&\bar{B}_{s\alpha\beta} \left(2G^{II} \mu_{2\alpha\beta} \bar{\beta}\right) J_0(\mu_{2\alpha\beta} r) - \bar{B}_{s\alpha\beta} \left(2G^{II} \mu_{2\alpha\beta} \bar{\beta}\right) \cdot \\
&\qquad\qquad \left.\left. Y_0(\mu_{2\alpha\beta} r) \Big\} dr \right) \right\| \sin(\omega_n t) \qquad (287)
\end{aligned}
$$

where

$$\mu_{1\alpha\beta}^{2} = \left(\frac{\omega_\alpha}{c_1^{II}}\right)^2 + \bar{\beta}^2 = \left[\left(\frac{\bar{c}_\alpha}{c_1^{II}}\right)^2 + 1\right]\bar{\beta}^2$$

$$\mu_{2\alpha\beta}^{2} = \left(\frac{\omega_\alpha}{c_2^{II}}\right)^2 + \bar{\beta}^2 = \left[\left(\frac{\bar{c}_\alpha}{c_2^{II}}\right)^2 + 1\right]\bar{\beta}^2 \qquad \text{(continued)}$$

$$\mu^2_{1\gamma\delta} = \left(\frac{\omega_\alpha}{c^I_1}\right)^2 + \bar{\beta}^2 = \left[\left(\frac{\bar{c}_\alpha}{c^I_1}\right)^2 + 1\right]\bar{\beta}^2$$

$$\mu^2_{2\gamma\delta} = \left(\frac{\omega_\alpha}{c^I_2}\right)^2 + \bar{\beta}^2 = \left[\left(\frac{\bar{c}_\alpha}{c^I_2}\right)^2 + 1\right]\bar{\beta}^2 \tag{288}$$

and

$$\bar{A}_{7\alpha\beta} = \bar{A}_{8\alpha\beta} = \bar{B}_{7\alpha\beta} = \bar{B}_{8\alpha\beta} = \bar{C}_{7\alpha\beta} = \bar{D}_{7\alpha\beta} = 0 \tag{289}$$

From equation (300), we have

$$\frac{2(2n - 1)\pi}{T} = \omega_\alpha = \omega_n \quad , \qquad n = 1,2,3... \tag{290}$$

Applying boundary conditions of displacements and stresses at interface and outer surface (equations (45), (46), (229) through (231) and (234)), we get the following relationships between coefficients

$$\bar{A}_{6\alpha\beta} = \frac{|\bar{N}_{1ij}|}{|\bar{\Delta}_{ij}|}\bar{A}_{5\alpha\beta} = \bar{M}_{1\alpha\beta}\bar{A}_{5\alpha\beta}$$

$$\bar{B}_{5\alpha\beta} = \frac{|\bar{N}_{2ij}|}{|\bar{\Delta}_{ij}|}\bar{A}_{5\alpha\beta} = \bar{M}_{2\alpha\beta}\bar{A}_{5\alpha\beta}$$

$$\bar{B}_{6\alpha\beta} = \frac{|\bar{N}_{3ij}|}{|\bar{\Delta}_{ij}|}\bar{A}_{5\alpha\beta} = \bar{M}_{3\alpha\beta}\bar{A}_{5\alpha\beta}$$

$$\bar{C}_{5\gamma\delta} = \frac{|\bar{N}_{4ij}|}{|\bar{\Delta}_{ij}|}\bar{A}_{5\alpha\beta} = \bar{M}_{4\alpha\beta}\bar{A}_{5\alpha\beta}$$

$$\bar{D}_{6\gamma\delta} = \frac{|\bar{N}_{5ij}|}{|\bar{\Delta}_{ij}|}\bar{A}_{5\alpha\beta} = \bar{M}_{5\alpha\beta}\bar{A}_{5\alpha\beta} \tag{291}$$

where determinants $|\bar{N}_{1ij}|$, $|\bar{N}_{2ij}|$, $|\bar{N}_{3ij}|$, $|\bar{N}_{4ij}|$, $|\bar{N}_{5ij}|$, and $|\bar{\Delta}_{ij}|$ are defined as follows

For $\bar{\Delta}_{ij}$ $(i,j = 1,\ldots,5)$,

$$\bar{\Delta}_{11} = \mu_{1\alpha\beta}\, Y_1(\mu_{1\alpha\beta}\, b)$$

$$\bar{\Delta}_{12} = -\bar{\beta}\, J_1(\mu_{1\alpha\beta}\, b)$$

$$\bar{\Delta}_{13} = -\bar{\beta}\, Y_1(\mu_{1\alpha\beta}\, b)$$

$$\bar{\Delta}_{14} = 0$$

$$\bar{\Delta}_{15} = 0 \tag{292}$$

$$\bar{\Delta}_{21} = \mu_{1\alpha\beta}\, Y_1(\mu_{1\alpha\beta}\, a)$$

$$\bar{\Delta}_{22} = -\bar{\beta}\, J_1(\mu_{2\alpha\beta}\, a)$$

$$\bar{\Delta}_{23} = -\bar{\beta}\, Y_1(\mu_{2\alpha\beta}\, a)$$

$$\bar{\Delta}_{24} = -\mu_{1\gamma\delta}\, J_1(\mu_{1\gamma\delta}\, a)$$

$$\bar{\Delta}_{25} = \bar{\beta}\, J_1(\mu_{2\gamma\delta}\, a) \tag{293}$$

$$\bar{\Delta}_{31} = \bar{\beta}\, Y_0(\mu_{1\alpha\beta}\, a)$$

$$\bar{\Delta}_{32} = -\mu_{2\alpha\beta}\, J_0(\mu_{2\alpha\beta}\, a)$$

$$\bar{\Delta}_{33} = -\mu_{2\alpha\beta}\, Y_0(\mu_{2\alpha\beta}\, a)$$

$$\bar{\Delta}_{34} = -\bar{\beta}\, J_0(\mu_{1\gamma\delta}\, a)$$

$$\bar{\Delta}_{35} = \mu_{2\gamma\delta}\, Y_0(\mu_{2\gamma\delta}\, a) \tag{294}$$

$$\overline{\Delta}_{41} = \left[\lambda^{II}\,\overline{\beta}^2 - \left(\lambda^{II} + 2G^{II}\right)\mu^2_{1\alpha\beta}\right] Y_0(\mu_{1\alpha\beta}\ a) + \frac{2G^{II}\,\mu_{1\alpha\beta}\ Y_1(\mu_{1\alpha\beta}\ a)}{a}$$

$$\overline{\Delta}_{42} = 2G^{II}\,\mu_{2\alpha\beta}\,\overline{\beta}\left[J_0(\mu_{2\alpha\beta}\ a) - \frac{J_1(\mu_{2\alpha\beta}\ a)}{\mu_{2\alpha\beta}\ a}\right]$$

$$\overline{\Delta}_{43} = 2G^{II}\,\mu_{2\alpha\beta}\,\overline{\beta}\left[Y_0(\mu_{2\alpha\beta}\ a) - \frac{Y_1(\mu_{2\alpha\beta}\ a)}{\mu_{2\alpha\beta}\ a}\right]$$

$$\overline{\Delta}_{44} = -\left[\lambda^{I}\,\overline{\beta}^2 - \left(\lambda^{I} + 2G^{I}\right)\mu^2_{1\gamma\delta}\right] J_0(\mu_{1\gamma\delta}\ a) - \frac{2G^{I}\,\mu_{1\gamma\delta}\ J_1(\mu_{1\gamma\delta}\ a)}{a}$$

$$\overline{\Delta}_{45} = -2G^{I}\,\mu_{2\gamma\delta}\,\overline{\beta}\left[J_0(\mu_{2\gamma\delta}\ a) - \frac{J_1(\mu_{2\gamma\delta}\ a)}{\mu_{2\gamma\delta}\ a}\right] \tag{295}$$

$$\overline{\Delta}_{51} = 2C^{II}\,\mu_{1\alpha\beta}\,\overline{\beta}\ Y_1(\mu_{1\alpha\beta}\ a)$$

$$\overline{\Delta}_{52} = -C^{II}\left(\mu^2_{2\alpha\beta} + \overline{\beta}^2\right) J_1(\mu_{2\alpha\beta}\ a)$$

$$\overline{\Delta}_{53} = -G^{II}\left(\mu^2_{2\alpha\beta} + \overline{\beta}^2\right) Y_1(\mu_{2\alpha\beta}\ a)$$

$$\overline{\Delta}_{54} = -2G^{I}\,\mu_{1\gamma\delta}\,\overline{\beta}\ J_1(\mu_{1\gamma\delta}\ a)$$

$$\overline{\Delta}_{55} = G^{I}\left(\mu^2_{2\gamma\delta} + \overline{\beta}^2\right) J_1(\mu_{2\gamma\delta}\ a) \tag{296}$$

For $\left|\bar{N}_{1\,ij}\right|$, the elements of the second to the fifth column are the same as the corresponding ones of $\left|\bar{\Delta}_{ij}\right|$, and the elements of the first column are as follows

$$(\bar{N}_1)_{11} = -\mu_{1\alpha\beta}\, J_1(\mu_{1\alpha\beta}\, b)$$

$$(\bar{N}_1)_{21} = -\mu_{1\alpha\beta}\, J_1(\mu_{1\alpha\beta}\, a)$$

$$(\bar{N}_1)_{31} = -\bar{\beta}\, J_0(\mu_{1\alpha\beta}\, a)$$

$$(\bar{N}_1)_{41} = -\left[\lambda^{II}\,\bar{\beta}^2 - \left(\lambda^{II} + 2G^{II}\right)\mu^2_{1\alpha\beta}\right] J_0(\mu_{1\alpha\beta}\, a) - \frac{2G^{II}\,\mu_{1\alpha\beta}\, J_1(\mu_{1\alpha\beta}\, a)}{a}$$

$$(\bar{N}_1)_{51} = -2G^{II}\,\mu_{1\alpha\beta}\,\bar{\beta}\, J_1(\mu_{1\alpha\beta}\, a) \qquad (297)$$

For $\left|\bar{N}_{2\,ij}\right|$, $\left|\bar{N}_{3\,ij}\right|$, $\left|\bar{N}_{4\,ij}\right|$, $\left|\bar{N}_{5\,ij}\right|$, the elements of the second, third, fourth, and fifth columns are, respectively, the same as equations (297) above, and the rest of their elements are the same as the corresponding elements in $\left|\bar{\Delta}_{ij}\right|$.

With $\bar{A}_{s\alpha\beta}$, $\bar{B}_{s\alpha\beta}$, $\bar{B}_{e\alpha\beta}$, $\bar{C}_{s\gamma\delta}$, $\bar{D}_{s\gamma\delta}$ defined as equations (291) to (297), we can rewrite equation (287) as follows

$$\frac{4P}{\pi}\left(\frac{1}{2n-1}\right) = -\left\|\sum_{\beta>0}^{\infty} \bar{A}_{s\alpha\beta}\left(\!\!\left(2\pi\int_a^b r\left\{\left[\left(\lambda^{II} + 2G^{II}\right)\bar{\beta}^2 - \lambda^{II}\mu^2_{1\alpha\beta}\right] J_0(\mu_{1\alpha\beta} r) + \bar{M}_{1\alpha\beta}\left[\left(\lambda^{II} + 2G^{II}\right)\bar{\beta}^2 - \lambda^{II}\mu^2_{1\alpha\beta}\right] Y_0(\mu_{1\alpha\beta} r) - \bar{M}_{2\alpha\beta}\left(2G^{II}\mu_{2\alpha\beta}\bar{\beta}\right)\cdot J_0(\mu_{2\alpha\beta} r) - \bar{M}_{3\alpha\beta}\left(2G^{II}\mu_{2\alpha\beta}\bar{\beta}\right) Y_0(\mu_{2\alpha\beta} r)\right\} dr\right.\right. +$$

$$+ \quad 2\pi \int_{o}^{a} r \left\{ \bar{M}_{4\alpha\beta} \left[\left(\lambda^{I} + 2G^{I} \right) \bar{\beta}^{2} - \lambda^{I} \mu_{1\gamma\delta}^{2} \right] J_0 (\mu_{1\gamma\delta} r) - \right.$$

$$\left. \bar{M}_{s\alpha\beta} \left(2G^{I} \mu_{s\gamma\delta} \bar{\beta} \right) J_0 (\mu_{s\gamma\delta} r) \right\} dr \Bigg] \Bigg\} \qquad (298)$$

In this equation, $\bar{\beta}$, $\mu_{1\alpha\beta}$, $\mu_{s\alpha\beta}$, $\mu_{1\gamma\delta}$, $\mu_{s\gamma\delta}$ are determined by equations (288) and the determinant $|\bar{d}_{ij}|$ (Appendix IV) with the use of

$$\omega_n = \frac{2(2n-1)\pi}{T} \quad , \quad n = 1,2,3,... \qquad (299)$$

Using the same technique in the representation of a function into a nonorthogonal eigenfunctions, we get

$$A_{s\alpha\beta} = -\frac{4P}{\pi} \Bigg\{ \bar{X}_1^2 \left(\frac{1}{2n-1} \right) 2\pi \int_o^a \left[\int_o^a \cdot \left\{ \bar{M}_{4\alpha\beta} \left[\left(\lambda^{I} + 2G^{I} \right) \bar{\beta}^{2} - \right. \right. \right.$$

$$\left. \left. \left. \lambda^{I} \mu_{1\gamma\delta}^{2} \right] J_0 (\mu_{1\gamma\delta} \bar{r}) - \bar{M}_{s\alpha\beta} \left(2G^{I} \mu_{s\gamma\delta} \bar{\beta} \right) J_0 (\mu_{s\gamma\delta} r) \right\} r' dr' \right] r dr +$$

$$\bar{X}_s^2 \left(\frac{1}{2n-1} \right) 2\pi \int_a^b \left[\int_a^b \left\{ \left[\left(\lambda^{II} + 2G^{II} \right) \bar{\beta}^{2} - \lambda^{II} \mu_{1\alpha\beta}^{2} \right] J_0 (\mu_{1\alpha\beta} r) + \right. \right.$$

$$\bar{M}_{1\alpha\beta} \left[\left(\lambda^{II} + 2G^{II} \right) \bar{\beta}^{2} - \lambda^{II} \mu_{1\alpha\beta}^{2} \right] Y_0 (\mu_{1\alpha\beta} r) -$$

$$\bar{M}_{s\alpha\beta} \left(2G^{II} \mu_{s\alpha\beta} \bar{\beta} \right) J_0 (\mu_{s\alpha\beta} r) - \bar{M}_{3\alpha\beta} \left(2G^{II} \mu_{s\alpha\beta} \bar{\beta} \right)$$

$$\left. \left. Y_0 (\mu_{s\alpha\beta} r) \right\} r' dr' \right] r dr \Bigg\} \quad \div$$

$$\Bigg\{ \bar{X}_1^2 \int_o^a \left[2\pi \int_o^a r' \left\{ \bar{M}_{4\alpha\beta} \left[\left(\lambda^{I} + 2G^{I} \right) \bar{\beta}^{2} - \lambda^{I} \mu_{1\gamma\delta}^{2} \right] J_0 (\mu_{1\gamma\delta} r) - \right. \right.$$

$$- \bar{M}_{s\alpha\beta}\left(2G^{I}\mu_{s\gamma\delta}\,\bar{\beta}\right) J_0(\mu_{s\gamma\delta}r)\Big\} \quad dr'\Big]^{a} rdr \;+\; \bar{X}_2^{a}\int_a^b \Big[2\pi \int_a^b r'\Big\{\Big[\Big(\lambda^{II} +$$

$$2G^{II}\Big)\bar{\beta}^{2} - \lambda^{II}\mu_{1\alpha\beta}^{2}\Big] J_0(\mu_{1\alpha\beta}r) + \bar{M}_{1\alpha\beta}\Big[\Big(\lambda^{II} + 2G^{II}\Big)\bar{\beta}^{2} -$$

$$\lambda^{II}\mu_{1\alpha\beta}^{2}\Big] Y_0(\mu_{1\alpha\beta}r) - \bar{M}_{s\alpha\beta}\left(2G^{II}\mu_{s\alpha\beta}\,\bar{\beta}\right) J_0(\mu_{s\alpha\beta}r) -$$

$$\bar{M}_{s\alpha\beta}\left(2G^{II}\mu_{s\alpha\beta}\,\bar{\beta}\right) Y_0(\mu_{s\alpha\beta}r)\Big\} \quad dr'\Big]^{a} rdr\Big) \qquad (300)$$

where $\bar{X}_1$ and $\bar{X}_2$ are defined as follows

$$\bar{X}_1^{a}\int_0^a r\Bigg(2\pi\int_a^b \Big\{\bar{M}_{4\alpha\beta}\Big[\Big(\lambda^{I} + 2G^{I}\Big)\bar{\beta}^{2} - \lambda^{I}\mu_{1\gamma\delta}^{2}\Big] J_0(\mu_{1\gamma\delta}r) -$$

$$\bar{M}_{s\alpha\beta}\left(2G^{I}\mu_{s\gamma\delta}\,\bar{\beta}\right) J_0(\mu_{s\gamma\delta}r)\Big]\Big\} r'dr' \cdot 2\pi\int_a^b \cdot\Big\{\bar{M}_{4\underline{\alpha\beta}}\Big[\Big(\lambda^{I} + 2G\Big)\underline{\bar{\beta}}^{2} -$$

$$\lambda^{I}\mu_{1\underline{\gamma\delta}}^{2}\Big] J_0(\mu_1\underline{\gamma\delta}r) - \bar{M}_{s\underline{\alpha\beta}}\left(2G^{I}\mu_{s\underline{\gamma\delta}}\,\underline{\bar{\beta}}\right) J_0(\mu_s\underline{\gamma\delta}r)\Big\} r'dr'\Bigg) dr +$$

$$\bar{X}_2^{a}\int_a^b r\Bigg(2\pi\int_a^b \Big\{\Big[\Big(\lambda^{II} + 2G^{II}\Big)\bar{\beta}^{2} - \lambda^{II}\mu_{1\alpha\beta}^{2}\Big] J_0(\mu_{1\alpha\beta}r) +$$

$$\bar{M}_{1\alpha\beta}\Big[\Big(\lambda^{II} + 2G^{II}\Big)\bar{\beta}^{2} - \lambda^{II}\mu_{1\alpha\beta}^{2}\Big] Y_0(\mu_{1\alpha\beta}r) -$$

$$\bar{M}_{s\alpha\beta}\left(2G^{II}\mu_{s\alpha\beta}\,\bar{\beta}\right) J_0(\mu_{s\alpha\beta}r) - \bar{M}_{s\alpha\beta}\left(2G^{II}\mu_{s\alpha\beta}\,\bar{\beta}\right)$$

$$Y_0(\mu_{s\alpha\beta}r)\Big\} r'dr' \cdot 2\pi\int_a^b \cdot\Big\{\Big[\Big(\lambda^{II} + 2G^{II}\Big)\underline{\bar{\beta}}^{2} - \lambda^{II}\mu_{1\underline{\alpha\beta}}^{2}\Big] J_0(\mu_1\underline{\alpha\beta}r) +$$

$$\bar{M}_{1\underline{\alpha\beta}}\Big[\Big(\lambda^{II} + 2G^{II}\Big)\underline{\bar{\beta}}^{2} - \lambda^{II}\mu_{1\underline{\alpha\beta}}^{2}\Big] Y_0(\mu_1\underline{\alpha\beta}r) -$$

$$- \bar{M}_{s\underline{\alpha}\underline{\beta}} \left(2G^{II} \mu_{s\underline{\alpha}\underline{\beta}}\ \underline{\bar{\beta}}\right) J_0 (\mu_{s\underline{\alpha}\underline{\beta}} r) -$$

$$\bar{M}_{3\underline{\alpha}\underline{\beta}} \left(2G^{II} \mu_{s\underline{\alpha}\underline{\beta}}\right) \underline{\bar{\beta}}\ \ Y_0 (\mu_{s\underline{\alpha}\underline{\beta}} r) \Big\} \ r'dr' \Big] \ dr \Big\} = 0 \qquad (301)$$

In the equations, $\underline{\alpha}$, $\underline{\beta}$, $\underline{\bar{\beta}}$, $\underline{\gamma}$, $\underline{\delta}$ indicate the values that are different from that of α, β, $\bar{\beta}$, γ, δ, respectively.

APPENDIX VIII

EVALUATION OF AN INTEGRAL

In this appendix are given the details on the evaluation of the integral

$$\tau(z,t) = \int_0^\infty \frac{\cos(\xi z)\,\sin\left(\xi\,\frac{at}{\sqrt{1+b^2\xi^2}}\right)d\xi}{\xi\left(1+b^2\xi^2\right)^{3/2}} \tag{302}$$

taking into account the development in series of sine, equation (302) can be written

$$\tau(z,t) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu+1)!}\,(at)^{2\nu+1}\int_0^\infty \frac{\xi^{2\nu}\cos\xi z}{\left(1+b^2\xi^2\right)^{2}\,(\nu+2)}\,d\xi \tag{303}$$

On the other hand, using the identity

$$\xi^{2\nu} = \sum_{\mu=0}^{\nu} A_\mu\left(1+b^2\xi^2\right)^\mu$$

with

$$A_\mu = (-1)^{\nu-\mu}\binom{\nu}{\mu}\,b^{-2\nu}$$

equation (303) is expressed as

$$\tau(z,t) = at\sum_{\nu=0}^{\infty} \frac{\left(\frac{at}{b}\right)^{2\nu}}{(2\nu+1)!}\sum_{\mu=0}^{\nu}(-1)^\mu\binom{\nu}{\mu}\int_0^\infty \frac{\cos\xi z\,d\xi}{\left(1+b^2\xi^2\right)^{2(\nu+2)-\mu}} \tag{304}$$

The following step is the evaluation of the integral that appears in equation (304),

$$\int_0^\infty \frac{\cos \xi z \, d\xi}{\left(1 + b^2 \xi^2\right)^{2(\nu+2)-\mu}} \tag{305}$$

for which we need to solve the complex integral

$$I = \int_0^\infty \frac{\cos \mu z}{\left(a^2 + z^2\right)^n} dz = \mathrm{Re} \int_0^\infty \frac{e^{i\mu z}}{\left(a^2 + z^2\right)^n} \tag{306}$$

where z is now the complex independent variable and μ and a are constants. Through development in the factors of the denominator, we obtain

$$I = \mathrm{Re} \frac{1}{(2ia)^n} \int_0^\infty \frac{e^{i\mu z} \, dz}{(z - ia)^n \left(1 + \frac{z - ia}{2ia}\right)^n} \tag{307}$$

taking into account that

$$\left(1 + \frac{z - ia}{2ia}\right)^{-n} = \sum_{m=0}^{\infty} \binom{-n}{m} \frac{(z - ia)^m}{(2ia)^m}$$

$$e^{i\mu z} = e^{-\mu a} \, e^{i\mu(z-ia)} = e^{-\mu a} \sum_{r=0}^{\infty} \frac{i^r}{r!} \mu^r (z - ia)^r$$

equation (307) becomes

$$I = \frac{e^{-\mu a}}{(2ia)^n} \int_0^\infty \frac{1}{(z - ia)^n} \sum_{m=0}^{\infty} \binom{-n}{m} \frac{(z - ia)^m}{(2ia)^m} \sum_{r=0}^{\infty} \frac{i^r}{r!} \mu^r (z - ia)^r \tag{308}$$

By putting $r + m - n = \sigma$, equation (308) is transformed into

$$I = \mathrm{Re}\, \frac{e^{-ua}}{(2ia)^n} \sum_{\sigma=-n}^{\infty} \int_0^{\infty} (z - ia)^{\sigma}\, dz \sum_{m=0}^{m+\sigma} \binom{-n}{m} \frac{(i)^{n+\sigma-m}\, \mu^{n+\sigma-m}}{(2ia)^m (n-m+\sigma)!} \tag{309}$$

Remembering that

$$\int_0^{\infty} (z - ia)^{\sigma}\, dz = \begin{cases} 2\pi i & \text{for } \sigma = -1 \\ 0 & \text{for } \sigma = 0 \end{cases}$$

equation (309) can be written in the following form:

$$I = \mathrm{Re}\, \frac{e^{-\mu a}}{(2a)^n}\, 2\pi \sum_{m=0}^{n-1} \binom{-n}{m} \frac{\mu^{n-1-m}}{(n-1-m)!}$$

$$= \frac{2\pi\, e^{-\mu a}}{(2a)^{2n-1}\, (n-1)!} \sum_{k=0}^{n-1} (2a\mu)^k\, \frac{(2n-k-2)!}{(n-k-1)!\; k!} \tag{310}$$

where $k = n - m - 1$.

Then, by using equations (306) and (310), we can write integral (305) in the following manner:

$$\int_0^{\infty} \frac{\cos \xi z}{(1+b^2\xi^2)^{2(\nu+2)-\mu}} = \frac{\pi e^{-z/b}}{b \cdot 4^{2\nu-\mu+3}} \cdot \frac{1}{(2\nu-\mu+3)!} \cdot$$

$$\sum_{k=0}^{2\nu-\mu+3} \left(\frac{2z}{b}\right)^k \frac{(4\nu-2\mu-k+6)!}{k!\,(2\nu-\mu-k+3)!} \tag{311}$$

By introducing equation (311) into equation (304), we finally obtain

$$\tau(z,t) = \int_0^\infty \frac{\cos(\xi z)\ \sin\left(\xi \dfrac{at}{\sqrt{1+b^2\xi^2}}\right) d\xi}{\xi\left(1+b^2\xi^2\right)^{3/2}}$$

$$= \frac{\pi}{4} e^{-z/b} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \nu!}{(2\nu+1)!} \left(\frac{at}{2b}\right)^{2\nu+1} \left(\frac{z}{b}\right)^\nu \sum_{\mu=0}^{\nu} \frac{(-1)^\mu \left(\frac{z}{b}\right)^\mu}{\mu!(\nu-\mu)!(\nu+\mu+3)!} \cdot$$

$$\sum_{k=-3}^{\nu+\mu} \left(\frac{b}{2z}\right)^k \frac{(\nu+\mu+k+6)!}{(3+k)!\ (\nu+\mu-k)!} \qquad (312)$$

APPENDIX IX

COMPUTER PROGRAMS FOR THE DETERMINATION OF EIGENFREQUENCIES AND WAVELENGTH IN A COMPOSITE ELEMENT

```
      PROGRAM FORCED
C
C        CIRCULAR OUTER BOUNDARY
C
      COMMON/BESSL/BESSELJ(2,2,3), BESSELY(2,2,3)
      COMMON/MU/FMU1AB, FMU2AB, FMU1GD, FMU2GD
      COMMON/IMAG/SIGN(4),IMAG(4)
      COMMON/INPUT/A,B,G1,G2,FLAMDA1,FLAMDA2,BETA,BETASQ,OMEGA,OMEGASQ
      COMMON/MAT/C(6,6),BB(6)
      COMMON/DATA/CI1SQ,CI2SQ,CII1SQ,CII2SQ
      COMMON/DATA2/CXAREA1,CXAREA2
      COMMON/COFF/A3AB,A4AB,B7AB,B8AB,C3GD,D7GD
      DIMENSION CC(6,6), SCRATCH(6), FMU(4)
      DIMENSION II(6)
      EQUIVALENCE(FMU(1), FMU1AB)
      EXTERNAL F1,F2,F3
      DATA(PI = 3.141592653)
C
      READ 993,VF,A,E1,E2,FNU1,FNU2,RHO1,RHO2,FL,OMEGA,BETA,DELTA
     1        ,FK11,FK12
      READ 701 , ITER
  993 FORMAT(8F10.5)
  700 FORMAT(4E20.5)
  701 FORMAT(5I5)
      FN = 1.
      J = 0      $      JFLAG = 1
      OMEGASQ = OMEGA * OMEGA
      B = A * SQRTF(1./VF)
      FLAMDA1 = E1*FNU1/((1.+FNU1)*(1.-2.*FNU1))
      FLAMDA2 = E2*FNU2/((1.+FNU2)*(1.-2.*FNU2))
      G1 = E1/(2.*(1.+FNU1))      $      G2 = E2/(2.*(1.+FNU2))
      CXAREA1 = PI * A*A
      CXAREA2 = PI * B*B  -  CXAREA1
      C I 1SQ =(2.*G1+FLAMDA1)/RHO1
      C I 2SQ = 2.*G2+FLAMDA2)/RHO2
      C II1SQ = G1/RHO1
      C II2SQ = G2/RHO2
C
      PRINT 600, A, E1, E2, B, FNU1, FNU2, VF, G1, G2, FL, RHO1, RHO2,
     *   FN, FLAMDA1, FLAMDA2, OMEGA, TOL
  600 FORMAT(3H1A  ,F20.5,10X,6HE1     ,E17.5,10X,6HE2      ,E17.5,/,
     *        3H0B  ,F20.5,10X,6HNU1    ,E17.5,10X,6HNU2     ,E17.5,/,
     *        3H0VF ,E20.5,10X,6HG   1  ,E17.5,10X,6HG   2   ,E17.5,/,
     *        3H0L  ,E20.5,10X,6HRHO 1  ,E17.5,10X,6HRHO 2   ,E17.5,/,
     *        3H0N  ,E20.5,10X,6HLAMDA1,E17.5,10X,6HLAMDA2,E17.5,/,
     *     6H0OMEGA,E17.5, / ,6H0TOL   ,E17.5, / ,4H3  J,7X,6HBETA  ,11X,
     *     5HDETRM,10X,11HFMU1AB (H2),9X,11HFMU2AB (K2),9X,11HFMU1GD (H1)
     *            9X,11HFMU2GD (K1), /)
C
    2 CALL MATRIX
C
      CALL NWMATINV(C, BB, SCRATCH, 6, 6, 0, DETRM, IDET)
      IF(IDET .EQ. 0) GO TO 20
      DETRM = DETRM * (10.**IDET)
C
   20 J = J+1
      PRINT 601, J, BETA , DETRM, (FMU(I), IMAG(I), I=1,4)
  601 FORMAT(I4, 2E17.8, 4( E17.7, A3), / )
C
      TOL = 1.E-6
      CALL ROOT (BETA, DETRM, TOL, DELTA , DIFF, JJ, JFLAG)
C
```

```
      IF(JJ) 70, 80, 89
   70 PRINT 602        $       GO TO 1
  602 FORMAT(////, 10X, 20HITERATIONS DIVERGING)
C
   80 IF(J .LT. ITER) GO TO 2
      PRINT 604, BETA, DIFF        $        GO TO 1
C
   89 PRINT 604, BETA, DIFF
  604 FORMAT(//, 7H BETA =,E17.8, 10X,7H DIFF =, E17.8)
C
      CALL MATRIX
C
      II = 1
      DO 90 I=1,6
         IF(I .EQ. 2)GO TO 90
         DO 89 J=2,6   $      JJ = J - 1
            CC(II,JJ) = C(I,J)
   89    CONTINUE
          BB(II) = -C(I,1)
         II = II + 1
   90 CONTINUE
C
      CALL NWMATINV(CC,BB,SCRATCH,6,5,1,DET,IDET)
C
      FM 6AB = BB(1)
      FM 7AB = BB(2)
      FM 8AB = BB(3)
      FM 9AB = BB(4)
      FM10AB = BB(5)
      BB(6) = 1.
C
      TEST = 1.E-4
      LIM = 20
      CALL SIMCON( A,B,TEST,LIM,AREA2,NO12,R2,F2)
      CALL SIMCON(0.,A,TEST,LIM,AREA3,NO13,R3,F3)
C
      AREA1 = PI*(((2.*FNU1*FK11/((1.+FNU1)*(1.-2.*FNU1)))+1.)*E1*A**2
     *    +((2.*FNU2*FK12/((1.+FNU2)*(1.-2.*FNU2)))+1.)*E2*(B**2-A**2))
      A3AB = -AREA1/((AREA2+AREA3)*SINF(BETA*FL))
      BB(6) = A3AB
C
      PRINT 605,(BB(J),J=1,5),A3AB
  605 FORMAT(/,'0M 6AB =',E20.5,10X,'M 7AB =',E20.5,10X,'M 8AB =',E20.5,
     *       /,'0M 9AB =',E20.5,10X,'M10AB =*,E20.5,10X,'A 3AB =',E20.5)
C
      A4AB = FM 6AB * A3AB
      B7AB = FM 7AB * A3AB
      B8AB = FM 8AB * A3AB
      C3GD = FM 9AB * A3AB
      D7GD = FM10AB * A3AB
C
      CALL STRWAVE
C
    1 END
```

```
      SUBROUTINE STRWAVE
C
C     CALCULATES DISPLACEMENT AND STRESSES USING CONSTANTS
C     OBTAINED BY PROGRAM SWAVE.
C
      COMMON/COFF/A3AB,A4AB,B7AB,B8AB,C3GD,D7GD
      COMMON/INPUT/A,B,G1,G2,FLAMDA1,FLAMDA2,BETA,BETASQ,OMEGA,OMEGASQ
      COMMON/IMAG/SIGN(4),IMAG(4)
      COMMON/MU/FMU1AB,FMU2AB,FMU1GD,FMU2GD
      EQUIVALENCE(SIGN(1),SIGN1),(SIGN(2),SIGN2),(SIGN(3),SIGN3),(SIGN(4
     1),SIGN4)
      PRINT 91
   91 FORMAT(1H1)
      BETASQ=BETA**2
  990 CONTINUE
      READ 702,R,Z
  702 FORMAT(2F10.5)
      IF(EOF,50) 950,960
  950 STOP
  960 CONTINUE
C
      IF(R.GT.A)GO TO 101
C
C     CALCULATE THE NECESSARY BESSEL FUNCTIONS
C
      Z1M1GD=Z1(FMU1GD*R,SIGN3)
      Z1M2GD=Z1(FMU2GD*R,SIGN4)
      Z0M1GD=Z0(FMU1GD*R,SIGN3)
      Z0M2GD=Z0(FMU2GD*R,SIGN4)
C
      XI1SUP1=-(C3GD*SIGN3*FMU1GD*Z1M1GD
     1          -D7GD*BETA*Z1M2GD)*SIN(BETA*Z)
C
      XI3SUP1=(C3GD*BETA*Z0M1GD
     1          +D7GD*FMU2GD*Z0M2GD)*COS(BETA*Z)
C
      S11SUP1=-(C3GD*((FLAMDA1*BETASQ+SIGN3*(FLAMDA1+2.*G1)*
     1          FMU1GD**2)*Z0M1GD
     2          -(2.*G1*FMU1GD*SIGN3)/R*Z1M1GD)
     3          -D7GD*(2.*G1*FMU2GD*BETA)*(Z0M2GD-Z1M2GD/(FMU2GD*R))
     4          )*SIN(BETA*Z)
C
      S22SUP1=-(C3GD*(FLAMDA1*(SIGN3*FMU1GD**2+BETASQ)*Z0M1GD
     1          +2.*G1*FMU1GD*SIGN3*Z1M1GD/R)
     2          -D7GD*((2.*G1*BETA)*Z1M2GD/R)
     3          )*SIN(BETA*Z)
C
      S33SUP1=-(C3GD*((FLAMDA1+2.*G1)*BETASQ+SIGN3*FLAMDA1*FMU1GD**2
     1          *Z0M1GD
     2          +D7GD*(2.*G1*FMU2GD*BETA)*Z0M2GD
     3         )*SIN(BETA*Z)
C
      S13SUP1=-(C3GD*(2.*SIGN3*G1*FMU1GD*BETA)*Z1M1GD
     1          +D7GD*G1*(SIGN4*FMU2GD**2-BETASQ)*Z1M2GD
     2          )*COS(BETA*Z)
C
C
      PRINT 805,R,Z
      PRINT 804,XI1SUP1,XI3SUP1
  804 FORMAT(' XI 1 FIBER = 'E15.5,5X,'XI 3 FIBER ='E15.5/)
      PRINT 807,S11SUP1,S22SUP1,S33SUP1,S13SUP1
C
```

```
      IF(R.LT.A)GO TO 290
  101 CONTINUE
C
C     DO SAME THING FOR RESIN
C
      Z1M1AB=Z1(FMU1AB*R,SIGN1)
      Z1M2AB=Z1(FMU2AB*R,SIGN2)
      ZOM1AB=ZO(FMU1AB*R,SIGN1)
      ZOM2AB=ZO(FMU2AB*R,SIGN2)
C
      W1M1AB=W1(FMU1AB*R,SIGN1)
      W1M2AB=W1(FMU2AB*R,SIGN2)
      WOM1AB=WO(FMU1AB*R,SIGN1)
      WOM2AB=WO(FMU2AB*R,SIGN2)
C
      XI1SUP2=-(A3AB*SIGN1*FMU1AB*Z1M1AB
     1         +A4AB*FMU1AB*W1M1AB-B7AB*BETA*Z1M2AB
     2         -B8AB*BETA*W1M2AB)*SIN(BETA*Z)
C
      XI3SUP2=(A3AB*BETA*ZOM1AB+A4AB*BETA*WOM1AB
     1         +B7AB*FMU2AB*ZOM2AB+B8AB*SIGN2*FMU2AB*WOM2AB)
     2         *COS(BETA*Z)
C
      S11SUP2=-(A3AB*((FLAMDA2*BETASQ+SIGN1*(FLAMDA2+2.*G2)*FMU1AB**2
     1         )*ZOM1AB-2.*G2*FMU1AB*SIGN1*Z1M1AB/R)
     2         +A4AB*((FLAMDA2*BETASQ+SIGN1*(FLAMDA2+2.*G2)*FMU1AB**2
     3         )*WOM1AB-2.*G2*FMU1AB*W1M1AB/R)
     4         -B7AB*(2.*G2*FMU2AB*BETA)*(ZOM2AB-Z1M2AB/(FMU2AB*R))
     5         -B8AB*(2.*G2*FMU2AB*BETA)*(SIGN2*WOM2AB-W1M2AB/(FMU2AB*R)
     6        ))*SIN(BETA*Z)
C
      S22SUP2=-(A3AB*(F   DA2*(SIGN1*FMU1AB**2+BETASQ)*ZOM1AB
     2         +2.*G2*FMU1AB*SIGN1*Z1 M 1AB/R)
     3         +A4AB*(FLAMDA2*(SIGN1*FMU1AB**2+BETASQ)*WOM1AB
     4         +2.*G2*FMU1AB*W1M1AB/R)
     5         -B7AB*(2.*G2*BETA*Z1M2AB/R)
     6         -B8AB*(2.*G2*BETA*W1M2AB/R))*SIN(BETA*Z)
C
      S33SUP2=-(A3AB*((FLAMDA2+2.*G2)*BETASQ+SIGN1*FLAMDA2*FMU1AB**2)
     1         *ZOM1AB
     2         +A4AB*((FLAMDA2+2.*G2)*BETASQ+SIGN1*FLAMDA2*FMU1AB**2)
     3         *WOM1AB
     4        +B7AB*2.*G2*FMU2AB*BETA*ZOM2AB
     4        +B8AB*2.*SIGN2*G2*FMU2AB*BETA*WOM2AB
     5         )*SIN(BETA*Z)
C
      S13SUP2=-(A3AB*2.*SIGN1*G2*FMU1AB*BETA*Z1M1AB
     1         +A4AB*2.*G2*FMU1AB*BETA*W1M1AB
     2         +B7AB*G2*(SIGN2*FMU2AB**2-BETASQ)*Z1M2AB
     3         +B8AB*G2*(SIGN2*FMU2AB**2-BETASQ)*W1M2AB
     4         )*COS(BETA*Z)
C
C
C     PRINT RESULTS FROM RESIN
C
      PRINT 805,R,Z
  805 FORMAT(//20X,' R = ',F15.8,5X,' Z = ',F15.2/20X,42(1H-)  ,//)
C
      PRINT 806,XI1SUP2,XI3SUP2
  806 FORMAT(' XI 1 RESIN =',F15.5,5X,'XI 3 RESIN =',F15.5/)
C
      PRINT 807,S11SUP2,S22SUP2,S33SUP2,S13SUP2
```

```
 807 FORMAT(' SIGMA 11 =',E15.5,5X,'SIGMA 22 =',E15.5/' SIGMA 33 =',E15
    1.5,5X,'SIGMA 13 =',E15.5/)
C
C
C
 290 CONTINUE
     GO TO 999
     END
```

```
      FUNCTION BEX(ARG,SIGN)
C
C     CALCULATES J AND Y OF ARG IF SIGN=1.
C     CALCULATES I AND K OF ARG IF SIGN=-1.
C
      DIMENSION ANS(5)
C
      ENTRY Z1
C
      IF(SIGN)10,10,20
   20 CALL BESSEL(ARG,1,ANS,1,0)
      BEX=ANS(2)
      RETURN
C
   10 CALL BESSEL(ARG,1,ANS,3,0)
      BEX=ANS(2)
      RETURN
C
      ENTRY Z0
C
      IF(SIGN)30,30,40
   40 CALL BESSEL(ARG,1,ANS,1,0)
      BEX=ANS(1)
      RETURN
C
   30 CALL BESSEL(ARG,1,ANS,3,0)
      BEX=ANS(1)
      RETURN
C
      ENTRY W1
      IF(SIGN)50,50,60
   60 CALL BESSEL(ARG,1,ANS,2,0)
      BEX=ANS(2)
      RETURN
C
   50 CALL BESSEL(ARG,1,ANS,4,0)
      BEX=ANS(2)
      RETURN
C
      ENTRY W0
C
      IF(SIGN)70,70,80
   70 CALL BESSEL(ARG,1,ANS,2,0)
      BEX=ANS(1)
      RETURN
C
   80 CALL BESSEL(ARG,1,ANS,4,0)
      BEX=ANS(1)
      RETURN
      END
```

```
      FUNCTION F1(R)
C
      COMMON/INPUT/A,B,G1,G2,FLAMDA1,FLAMDA2,BETA,BETASQ,OMEGA,OMEGASQ
      COMMON/MU/FMU1AB, FMU2AB, FMU1GD, FMU2GD
      COMMON/MUSQ/SOFMU1AB,SOFMU2AB,SOFMU1GD,SOFMU2GD
      COMMON/DATA2/CXAREA1,CXAREA2
      COMMON/IMAG/SIGN1,SIGN2,SIGN3,SIGN4,IMAG(4)
      COMMON/MAT/C(6,6),M6AB,M7AB,M8AB,M9AB,M10AB,A3AB
      TYPE REAL          M6AB,M7AB,M8AB,M9AB,M10AB
      DIMENSION ANSZ(4),ANSW(4)
      DATA (PI = 3.141592653)
C
C
      F1 = 1./PI
      RETURN
C
C
                     ENTRY   F 2
C
      IF(R .EQ. 0.0) GO TO 20
      ARG1 = FMU1AB*R
      ARG2 = FMU2AB*R
      IF(SIGN1 .LT. 0.0)GO TO 1
         CALL BESSEL(ARG1,0,ANSZ,1,0)
      FF2 = (FLAMDA2+2.*G2)*BETASQ+FLAMDA2*SOFMU1AB
         CALL BESSEL(ARG1,0,ANSW,2,0)
      FF2 = FF2 * (ANSZ(1)+M6AB*ANSW(1))
      GO TO 2
    1    CALL BESSEL(ARG1,0,ANSZ,3,0)
         CALL BESSEL(ARG1,0,ANSW,4,0)
      FF2 = FF2 * (ANSZ(1)-M6AB*2.*ANSW(1))
C
      IF(SIGN2 .LT. 0.0)GO TO 3
    2 FF = 2.*G2*FMU2AB*BETA
         CALL BESSEL(ARG2,0,ANSZ,1,0)
         CALL BESSEL(ARG2,0,ANSW,2,0)
      FF2 = FF2+FF*(M7AB+M8AB)
      GO TO 4
C
    3    CALL BESSEL(ARG2,1,ANSW,3,0)
      FF2 = FF2 - FF*M8AB*ANSW(1)
C
    4 F1 = A3AB *          FF2 * R * 2. * PI
      RETURN
C
C
                     ENTRY   F 3
C
      IF(R .EQ. 0.0) GO TO 20
      ARG3 = FMU1GD * R
      IF(SIGN3 .LT. 0.0)GO TO 11
         CALL BESSEL(ARG3,0,ANSZ,1,0)
      GO TO 12
   11    CALL BESSEL(ARG3,0,ANSZ,3,0)
   12 FF3 = M9AB*((FLAMDA1+2.*G1)*BETASQ + FLAMDA1*
     *      SOFMU1GD)*ANSZ(1)
C
      IF(SIGN4 .LT. 0)GO TO 13
      ARG4 = FMU2GD * R
         CALL BESSEL(ARG4,0,ANSZ,1,0)
      FF3 = FF3 + M10AB*2.*G1*FMU2GD*BETA*ANSZ(1)
   13 F1 = A3AB*         FF3  * R * 2. * PI
```

```
      RETURN
C
C
   20 F1 = 0.0
      RETURN
C
      END
```

```
      SUBROUTINE MATRIX
      COMMON/BESSL/BESSELJ(2,2,3), BESSELY(2,2,3)
      COMMON/INPUT/A,B,G1,G2,FLAMDA1,FLAMDA2,BETA,BETASQ,OMEGA,OMEGASQ
      COMMON/MU/FMU1AB, FMU2AB, FMU1GD, FMU2GD
      COMMON/MUSQ/SQFMU1AB,SQFMU2AB,SQFMU1GD,SQFMU2GD
      COMMON/IMAG/SIGN(4),IMAG(4)
      COMMON/MAT/C(6,6),BB(6)
      COMMON/DATA/CI1SQ,CI2SQ,CII1SQ,CII2SQ
      DIMENSION FMU(4),SQFMU(4)
      EQUIVALENCE (FMU,FMU1AB), (SQFMU,SQFMU1AB)
C
C
      BETASQ = BETA * BETA
C
      FMU(1) = OMEGASQ/CI 2SQ - BETASQ
      FMU(2) = OMEGASQ/CII2SQ - BETASQ
      FMU(3) = OMEGASQ/CI 1SQ - BETASQ
      FMU(4) = OMEGASQ/CII1SQ - BETASQ
C
      DO 6  I = 1,4
      IF(FMU(I)) 3,3,4
    3 IMAG(I) = 3H  I     $     SIGN(I) = -1.0     $     FMU(I) = -FMU(I)
      K = I
      GO TO 5
    4 IMAG(I) = 3H          $     SIGN(I) =  1.0
    5 SQFMU(I) = SIGN(I) * FMU(I)
        FMU(I) = SQRTF    (FMU(I))
    6 CONTINUE
      IF(K .EQ. 4) GO TO 2
C
      CALL BESLFUN
C
   11 C(1,1) = FMU1AB*BESSELJ(2,1,2)
   12 C(1,2) = FMU1AB*BESSELY(2,1,2)
   13 C(1,3) = -BETA*BESSELJ(2,2,2)
   14 C(1,4) = -BETA*BESSELY(2,2,2)
   15 C(1,5) = C(1,6) = C(2,5) = C(2,6) = 0.
C
   21 C(2,1) = 2.*FMU1AB*BETA*BESSELJ(2,1,2)
   22 C(2,2) = 2.*FMU1AB*BETA*BESSELY(2,1,2)
   23 C(2,3) = (SQFMU2AB-BETASQ)*BESSELJ(2,2,2)
   24 C(2,4) = (SQFMU2AB-BETASQ)*BESSELY(2,2,2)
C
   31 C(3,1) = FMU1AB*BESSELJ(2,1,1)
   32 C(3,2) = FMU1AB*BESSELY(2,1,1)
      IF(SIGN(2))132,132,33
  132 C(3,3) = 0.0 $ GO TO 34
   33 C(3,3) = -BETA*BESSELJ(2,2,1)
   34 C(3,4) = -BETA*BESSELY(2,2,1)
   35 C(3,5) =-FMU1GD*BESSELJ(2,1,3)
   36 C(3,6) = BETA*BESSELJ(2,2,3)
C
   41 C(4,1) = BETA*BESSELJ(1,1,1)
   42 C(4,2) = BETA*BESSELY(1,1,1)
   43 C(4,3) = FMU2AB*BESSELJ(1,2,1)
   44 C(4,4) = FMU2AB*BESSELY(1,2,1)
   45 C(4,5) = -BETA*BESSELJ(1,1,3)
   46 C(4,6) = -FMU2GD*BESSELJ(1,2,3)
C
   51 C(5,1) = (FLAMDA2*BETASQ+(FLAMDA2+2.*G2)*SQFMU1AB)*
     *          BESSELJ(1,1,1)-2.*G2*FMU1AB*BESSELJ(2,1,1)/A
   52 C(5,2) = (FLAMDA2*BETASQ+(FLAMDA2+2.*G2)*SQFMU1AB)*
```

```
     *         BESSELY(1,1,1)-2.*G2*FMU1AB*BESSELY(2,1,1)/A
  53 C(5,3) = -2.*G2*FMU2AB*BETA*(BESSELJ(1,2,1)-BESSELJ(2,2,1)/
     *         (FMU2AB*A))
  54 C(5,4) = -2.*G2*FMU2AB*BETA*(BESSELY(1,2,1)-BESSELY(2,2,1)/
     *         (FMU2AB*A))
  55 C(5,5) = -(FLAMDA1*BETASQ+(FLAMDA1+2.*G1)*SQFMU1GD)*
     *         BESSELJ(1,1,3)+2.*G1*FMU1GD*BESSELJ(2,1,3)/A
  56 C(5,6) = 2.*G1*FMU2GD*BETA*(BESSELJ(1,2,3)-BESSELJ(2,2,3)/
     *         (FMU2GD*A))
C
  61 C(6,1) = 2.*G2*FMU1AB*BETA*BESSELJ(2,1,1)
  62 C(6,2) = 2.*G2*FMU1AB*BETA*BESSELY(2,1,1)
     IF(SIGN(2))162,162,63
 162 C(6,3) = 0.0 $ GO TO 64
  63 C(6,3) = G2*(SQFMU2AB-BETASQ)*BESSELJ(2,2,1)
  64 C(6,4) = G2*(SQFMU2AB-BETASQ)*BESSELY(2,2,1)
  65 C(6,5) =-G1*2.*FMU1GD*BETA*BESSELJ(2,1,3)
  66 C(6,6) =-G1*(SQFMU2GD-BETASQ)*BESSELJ(2,2,3)
   1 RETURN
   2 PRINT 500,BETA  $.   REMOVE THIS CARD TO ALLOW IMAGINARY ARGUMENTS
     STOP1           $.   REMOVE THIS CARD TO ALLOW IMAGINARY ARGUMENTS
 500 FORMAT(' BETA ='E17.5)  $. REVOVE THIS CARD
     END
```

```
      SUBROUTINE BESLFUN
      COMMON/BESSL/BESSELJ(2,2,3),BESSELY(2,2,3)
      COMMON/INPUT/A,B,G1,G2,FLAMDA1,FLAMDA2,BETA,BETASQ,OMEGA,OMEGASQ
      COMMON/MU/FMU1AB,FMU2AB,FMU1GD,FMU2GD
      COMMON/IMAG/SIGN1,SIGN2,SIGN3,SIGN4,IMAG(4)
      DIMENSION ANS(2)
      DATA(PI = 3.141592653)
C
      X1 = FMU1AB * A      $      XX1 = FMU2AB * A
      X2 = FMU1AB * B      $      XX2 = FMU2AB * B
      X3 = FMU1GD * A      $      XX3 = FMU2GD * A
C
      IF(SIGN1)1,1,5
    1 CALL BESSEL(X 1,1,ANS,3,0)
                    BESSELJ(1,1,1) =  ANS(1)
                    BESSELJ(2,1,1) = -ANS(2)
      CALL BESSEL(X 1,1,ANS,4,0)
                    BESSELY(1,1,1) = -ANS(1)
                    BESSELY(2,1,1) = -ANS(2)
      CALL BESSEL(X 2,1,ANS,3,0)
                    BESSELJ(2,1,2) =  ANS(2)
      CALL BESSEL(X 2,1,ANS,4,0)
                    BESSELY(2,1,2) =  ANS(2)
      GO TO 6
C
C
    5 CALL BESSEL(X 1,1,ANS,1,0)
                    BESSELJ(1,1,1) =  ANS( )
                    BESSELJ(2,1,1) =  ANS(2)
      CALL BESSEL(X 1,1,ANS,2,0)
                    BESSELY(1,1,1) =  ANS(1)
                    BESSELY(2,1,1) =  ANS(2)
      CALL BESSEL(X 2,1,ANS,1,0)
                    BESSELJ(2,1,2) =  ANS(2)
      CALL BESSEL(X 2,1,ANS,2,0)
                    BESSELY(2,1,2) =  ANS(2)
C
    6 IF(SIGN2)7,7,10
C
    7 CALL BESSEL(XX1,1,ANS,3,0)
                    BESSELJ(1,2,1) =  0.0
                    BESSELJ(2,2,1) = -ANS(2)
                    BESSELY(1,2,1) = -ANS(1)
                    BESSELY(2,2,1) = -ANS(2)
                    BESSELJ(2,2,2) =  0.0
      CALL BESSEL(XX2,1,ANS,3,0)
                    BESSELY(2,2,2) = -ANS(2)
      GO TO 11
C
   10 CALL BESSEL(XX1,1,ANS,1,0)
                    BESSELJ(1,2,1) =  ANS(1)
                    BESSELJ(2,2,1) =  ANS(2)
      CALL BESSEL(XX1,1,ANS,2,0)
                    BESSELY(1,2,1) =  ANS(1)
                    BESSELY(2,2,1) =  ANS(2)
      CALL BESSEL(XX2,1,ANS,1,0)
                    BESSELJ(2,2,2) =  ANS(2)
      CALL BESSEL(XX2,1,ANS,2,0)
                    BESSELY(2,2,2) =  ANS(2)
C
   11 IF(SIGN3)12,12,15
C
```

```
   12 CALL BESSEL(X 3,1,ANS,3,0)
                        BESSELJ(1,1,3) =  ANS(1)
                        BESSELJ(2,1,3) = -ANS(2)
      GO TO 16
C
   15 CALL BESSEL(X 3,1,ANS,1,0)
                        BESSELJ(1,1,3) =  ANS(1)
                        BESSELJ(2,1,3) =  ANS(2)
C
   16 IF(SIGNA)17,17,20
C
   17                   BESSELJ(1,2,3) =  0.0
                        BESSELJ(2,2,3) =  0.0
      RETURN
C
   20 CALL BESSEL(XX3,1,ANS,1,0)
                        BESSELJ(1,2,3) =  ANS(1)
                        BESSELJ(2,2,3) =  ANS(2)
C
      RETURN
C
      END
```

```
      SUBROUTINE ROOT(X, Y, TOL, DEL, DIFF, IFLAG, JFLAG)
C
C        THIS SUBROUTINE WILL FIND A ROOT BY FALSE POSITION
C
      IFLAG = 0
      GO TO (10, 20, 30) JFLAG
C
   10 X1 = X          $    Y1 = Y
      JFLAG = 2
   15 X = Y + DEL     $    RETURN
C
   20 X2 = Y          $    Y2 = Y
      IF (Y1 * Y2) 25, 50, 21
   21 X1 = X2         $    Y1 = Y2      $    GO TO 15
   25 JFLAG = 3
   26 X = X2 - ((X2-X1)/(Y2-Y1)) * Y2   $    RETURN
C
   30 X3 = X          $    Y3 = Y
C
      DIFF = ABSF(X3 - X2)
      IF(DIFF .LT. TOL) GO TO 50
      DIFF = ABSF(X3 - X1)
      IF(DIFF .LT. TOL) GO TO 50
C
   31 IF(Y1 * Y3) 32, 50, 33
   32 X2 = X3         $    Y2 = Y3      $    GO TO 26
   33 IF(Y2 * Y3) 34, 50, 40
   34 X1 = X3         $    Y1 = Y3      $    GO TO 26
C
   40 IFLAG = -1      $    RETURN
C
   50 IFLAG = 1       $    RETURN       $    END
```

```
      SUBROUTINE BESSEL (X,N,VALUE,INDEX,MM)
      DIMENSION VALUE(41),BJ(100),F(100)
      DIMENSION D1(2),D2(2)
      EQUIVALENCE (D1(2),BJ(1)),(D2(2),F(1))
      DATA (PI=3.14159265)
      SINH(X)=0.5*(EXPF(X)-EXPF(-X))
      IKL=0
      IF (X) 10,10,9
    9 IK=0.5*X
      K=N+20+IK
      IF(K-100) 300,299,299
  299 PRINT 298
  298 FORMAT(66H ARGUMENT PLUS ORDER TOO LARGE. INCREASE DIMENSIONS AND
     1RUN AGAIN.)
      RETURN
  300 CONTINUE
      Z=1.0/X
      GOTO(40,40,60,60,80,90,100,110),INDEX
   10 PRINT 8,X
    8 FORMAT (1X,63HSUBROUTINE BESSEL DOES NOT WORK FOR X=0 OR LESS THAN
     1 0. HERE X= E15.8)
      RETURN
   40 BJ(K+1)=0.0
      BJ(K)=10.0E-30
      L=K
   11 FL=L
      BJ(L-1)=2.0*FL*Z*BJ(L)-BJ(L+1)
      IF(L-1) 12,12,13
   13 L=L-1
      GO TO 11
   12 SUM=0.0
      IK=K
      DO 14 I=2,IK,2
   14 SUM=SUM+BJ(I)
      C=1.0/(BJ(IKL)+2.0*SUM)
      GO TO(42,43),INDEX
   43 DO 45 I=IKL,K
   45 BJ(I)=C*BJ(I)
      GO TO 54
   42 DO 41 I=IKL,N
   41 VALUE(I+1)=C*BJ(I)
      RETURN
   50 GO TO 40
   54 EC=.5772156649
      SUM=0.0
      IK=K/2
      DO 15 I=1,IK
      FI=I
   15 SUM=SUM+((-1.0)**(I-1)*BJ(2*I))/FI
      F(IKL)=(2.0/PI)*(BJ(IKL)*(LOGF(0.5*X)+EC)+2.0*SUM)
      F(1)=(BJ(1)*F(IKL)-2.0/(PI*X))/BJ(IKL)
      IN=N-1
      DO 16 I=1,IN
      FI=I
   16 F(I+1)=2.0*FI*Z*F(I)-F(I-1)
      DO 51 I=IKL,N
   51 VALUE(I+1)=F(I)
      RETURN
   60 BJ(K+1)=0.
      BJ(K)=10.E-30
      L=K
   10 FL=L
```

```
      BJ(L-1)=2.*FL*Z*BJ(L)+BJ(L+1)
      IF(L-1)17,17,18
   18 L=L-1
      GO TO 19
   17 SUM=0.0
      DO 21 I=1,K
   21 SUM=SUM+BJ(I)
      C=(EXPF(X))/(BJ(IKL)+2.*SUM)
      GO TO(1,1,61,62),INDEX
    1 PRINT 1000
      RETURN
 1000 FORMAT(40H CARD MISSING. IGNORE RESULTS.                )
   62 DO 64 I=IKL,1
   64 BJ(I)=C*BJ(I)
      GO TO 71
   61 DO 65 I=IKL,N
   65 VALUE(I+1)=C*BJ(I)
      RETURN
   70 GO TO 60
   71 IF(5.0-X) 120,121,121
  121 M=10
      GO TO 122
  120 M=5
  122 F(IKL)=GAUSS(0.0,1.0,M,X)
      F(1)=(1./X-F(IKL)*BJ(1))/BJ(IKL)
      IN=N-1
      DO 22 I=1,IN
      FI=I
   22 F (I+1)=2.0*FI*Z*F (I)+F (I-1)
      DO 72 I=IKL,N
   72 VALUE(I+1)=F(I)
      RETURN
   80 F(K+1)=0.0
      F (K)=10.0E-30
      L=K
   25 FL=L
      F (L-1)=(2.0*FL+1.0)*Z*F (L)-F (L+1)
      IF (L-1)  23,23,24
   24 L=L-1
      GO TO 25
   23 C=SINF(X)/(X*F(IKL))
      IF(MM)200,200,201
  201 SQ=SQRTF(2.*X/PI)
      DO 202 I=IKL,N
  202 VALUE(I+1)=C*SQ*F(I)
      RETURN
  200 DO 26 I=IKL,N
   26 VALUE(I+1)=C*F(I)
      RETURN
   90 F(IKL)=-COSF(X)/X
      F (1)=-SINF(X)/X-COSF(X)/X**2
      IN=N-1
      DO 27 I=1,IN
      FI=I
   27 F(I+1)=(2.*FI+1.0)*Z*F(I)-F(I-1)
      IF(MM) 203,203,204
  204 SQ=SQRTF(2.*X/PI)
      DO 205 I=IKL,N
  205 VALUE(I+1)=F(I)*SQ*(-1.)**(-I-1)
      RETURN
  203 DO 91 I=IKL,N
   91 VALUE(I+1)=F(I)
```

```
      RETURN
  100 F (K+1)=0.0
      F (K)=10.0E-30
      L=K
   30 FL=L
      F (L-1)=(2.0*FL+1.0)*Z*F (L)+F (L+1)
      IF (L-1) 28,28,29
   29 L=L-1
      GO TO 30
   28 C=SINH(X)/(X*F(IKL))
      IF(MM) 206,206,207
  207 SQ=SQRTF(2.*X/PI)
      DO 208 I=IKL,N
  208 VALUE(I+1)=C*SQ*F(I)
      RETURN
  206 DO 101 I=IKL,N
  101 VALUE(I+1)=C*F(I)
      RETURN
  110 F(IKL)=0.5*(-PI*EXPF(-X)/X)
      F(1)=0.5*(PI*(X+1.0)*EXPF(-X))/X**2
      IN=N-1
      DO 31 I=1,IN
      FI=I
   31 F (I+1)=F (I-1)-(2.0*FI+1.0)*Z*F (I)
      IF(MM) 209,209,210
  210 SQ=SQRTF(2.*X/PI)
      DO 211 I=IKL,N
  211 VALUE(I+1)=F(I)*SQ*(-1.)**(-I-1)
      RETURN
  209 DO 111 I=IKL,N
  111 VALUE(I+1)=F(I)
      RETURN
      END
```

```
      FUNCTION GAUSS(G,B,M,X)
      DIMENSION R(10),U(10)
      U(1)=+.0744371695
      U(2)=-.0744371695
      U(3)=+.2166976971
      U(4)=-.2166976971
      U(5)=+.3397047841
      U(6)=-.3397047841
      U(7)=+.4325316833
      U(8)=-.4325316833
      U(9)=+.4869532643
      U(10)=-.4869532643
      R(1)=+.1477621124
      R(2)=+.1477621124
      R(3)=+.1346333597
      R(4)=+.1346333597
      R(5)=.1095431813
      R(6)=.1095431813
      R(7)=+.07472567458
      R(8)=+.07472567458
      R(9)=+.03333567215
      R(10)=+.03333567215
      A=G
      FN=M
      P=(B-A)/FN
      GAUSS=0.0
      DO 1 J=1,M
      C=0.0
      Y=A+P
      DO 2 I=1,10
      D=(R(I)*GAUSSF((Y-A)*U(I)+(A+Y)/2.0,X))*(Y-A)
    2 C=D+C
      A=Y
    1 GAUSS=GAUSS +C
      RETURN
      END

      FUNCTION GAUSSF(U,X)
      COSH(X)=0.5*(EXPF(X)+EXPF(-X))
      IF(1.0/U-710.0)123,124,124
  124 GAUSSF=0.0
      RETURN
  123 GAUSSF= (EXPF(-X*COSH(1.0/U-1.0)))/U**2
      END
```

```
      SUBROUTINE SIMCON(X1,XEND,TEST,LIM,AREA,NOI,R,F)                  SIMCON
C
C *** INTEGRATES THE EXTERNAL FUNCTION F BETWEEN THE LIMITS X1 AND XEND.
C *** IT SUCCESSIVELY HALVES THE INTERVAL UNTIL THE ERROR IS LESS THAN TEST.
C
      NOI=0                                                             SIMCON 4
      R1=10.0                                                           SIMCON 5
      ODD=0.0                                                           SIMCON 6
      INT=1                                                             SIMCON 7
      V=1.0                                                             SIMCON 8
      EVEN=0.0                                                          SIMCON 9
      AREA1=0.0                                                         SIMCON10
 19   ENDS=F(X1)+F(XEND)                                                SIMCON11
 2    H=(XEND-X1)/V                                                     SIMCON12
      ODD=EVEN+ODD                                                      SIMCON13
      X=X1+H/2.                                                         SIMCON14
      EVEN=0.0                                                          SIMCON15
      DO 3 I=1,INT                                                      SIMCON16
 21      EVEN=EVEN+F(X)                                                 SIMCON17
         X=X+H                                                          SIMCON18
 3    CONTINUE                                                          SIMCON19
 31   AREA=(ENDS+4.0*EVEN+2.0*ODD)*H/6.0                                SIMCON20
      NOI=NOI+1                                                         SIMCON21
 34   R=ABSF((AREA1-AREA)/AREA)                                         SIMCON22
      IF(NOI-LIM)341,35,35                                              SIMCON23
 341  IF(R-TEST)35,35,4                                                 SIMCON24
 35   RETURN                                                            SIMCON25
 4    AREA1=AREA                                                        SIMCON26
 46   INT=2*INT                                                         SIMCON27
      V=2.0*V                                                           SIMCON28
      GO TO 2                                                           SIMCON29
      END                                                               SIMCON30
```

```
      SUBROUTINE NWMATINV(A,B,INDEX,NMAX,N,M,DETERM,IDET)
      DIMENSION A(NMAX,1),B(NMAX,1),INDEX(1)
      EQUIVALENCE(IROW,JROW,IR),(ICOLUM,JCOLUM,IC)
      EQUIVALENCE(AMAX,T,SWAP,IAMAX),(PIVOT,TEMP,ITEMP)
      DATA (MINUS=60000000000000000B)
C**   INITIALIZATION.
      DETERM=1.0
      IDET = 0
      DO 20 J=1,N
   20 INDEX(J)=MINUS
      DO 550 I=1,N
C**   SEARCH FOR ELEMENT OF LARGEST MAGNITUDE.
      AMAX=0.0
      DO 105 J1=1,N
      IF(-INDEX(J1)) 105,105,60
   60 DO 100 K1=1,N
      IF(-INDEX(K1)) 100,100,80
   80 TEMP=A(K1,J1)
      IF(TEMP)83,100,82
   82 TEMP=-TEMP
   83 IF(-ITEMP-IAMAX) 100,100,84
   84 AMAX=-TEMP
      IR=K1
      IC=J1
  100 CONTINUE
  105 CONTINUE
      IF(AMAX)120,115
  115 DETERM=0
      IDET = 0
      RETURN
  120 IROW=IR
      ICOLUM=IC
      INDEX(ICOLUM)=INDEX(ICOLUM).AND..NOT.MINUS
      IF(.NOT.(IROW-ICOLUM))260,140
  140 DETERM=-DETERM
C**   EXCHANGE ROWS.
      DO 200 L=1,N
      SWAP=A(IROW,L)
      A(IROW,L)=A(ICOLUM,L)
  200 A(ICOLUM,L)=SWAP
      IF(.NOT.M)260,210
  210 DO 250 L=1, M
      SWAP=B(IROW,L)
      B(IROW,L)=B(ICOLUM,L)
  250 B(ICOLUM,L)=SWAP
C**   SAVE PIVOT INFORMATION. DO DETERMINANT.
  260 INDEX(I)=IROW*1000000000B+ICOLUM+INDEX(I)
      PIVOT    =A(ICOLUM,ICOLUM)
      DETERM=DETERM*PIVOT
C
C  **     KEEP DETERMINANT BETWEEN 1.0E-20 AND 1.0E+20
  270 DETERM1 = ABSF(DETERM)
      IF(DETERM1 .GT. 1.E-20) GO TO 275
      DETERM = DETERM * 1.E20
      IDET = IDET - 20
      GO TO 270
  274 DETERM1 = ABSF(DETERM)
  275 IF(DETERM1 .LT. 1.E20) GO TO 300
      DETERM = DETERM / 1.E20
      IDET = IDET + 20
      GO TO 274
  300 CONTINUE
```

```
C**    REDUCE LEADING COEF TO 1.
       A(ICOLUM,ICOLUM)=1.0
       DO 350 L=1,N
  350  A(ICOLUM,L)=A(ICOLUM,L)/PIVOT
       IF(.NOT.M)380,360
  360  DO 370 L=1,M
  370  B(ICOLUM,L)=B(ICOLUM,L)/PIVOT
C**    SUBSTITUTE FOR NTH VARIABLE.
  380  DO 550 L1=1,N
       IF(.NOT.(L1-ICOLUM)) 550,400
  400  T=A(L1,ICOLUM)
       A(L1,ICOLUM)=0.0
       DO 450 L=1,N
  450  A(L1,L)=A(L1,L)-A(ICOLUM,L)*T
       IF(.NOT.M)550,460
  460  DO 500 L=1,M
  500  B(L1,L)=B(L1,L)-B(ICOLUM,L)*T
  550  CONTINUE
C**    UNDO ROW EXCHANGES.
       L=N
       DO 710 L2=1,N
       JROW=INDEX(L)/1000000000B
       JROW=JROW.AND.77777B
       JCOLUM=INDEX(L).AND.77777B
       IF(.NOT.(JROW-JCOLUM))710,630
  630  DO 705 K=1,N
       SWAP=A(K,JROW)
       A(K,JROW)=A(K,JCOLUM)
  705  A(K,JCOLUM)=SWAP
  710  L=L-1
  740  RETURN
       END
```

```
      PROGRAM NULLMAT
      COMMON/BESSLL/BESL(4)
      COMMON/DATA/FH(2), FK(2), FLAMDA(2), FMU(2), IH(2), R, PI, GAMMA,
     *   GAMMASQ
      DIMENSION AA(6,6), CC(6), IMAG(4), SIGN(3), RHO(2)
      DATA(PI = 3.1415926531), (SIGN = 1., -1., -1.)
C
C          INPUT AND INITIALIZE DATA
C
    1 READ 700, VF, FN, OMEGA, DOMEGA, TOL, ITER
  700 FORMAT(5F15.5, I5)
C
      IF(EOF,50) 100, 20
   20 JFLAG = 1                    $      J = 0
      FL = 3.                      $      P = OMEGA
      A = .0025                    $      B = SQRTF(A*A/VF)
      E1 = 1.00 E 7                $      E2 = 3.80 E 5
      FNU1 = 0.2                   $      FNU2 = 0.35
      GAMMA = FN*PI/FL             $      GAMMASQ = GAMMA**2
      RHO(1) = 2.42754 E -4        $      RHO(2) = 1.19942 E -4
      FMU(1) = E1/(2.+2.*FNU1)
      FMU(2) = E2/(2.+2.*FNU2)
      FLAMDA(1) = FNU1*E1/((1.+FNU1)*(1.-2.*FNU1))
      FLAMDA(2) = FNU2*E2/((1.+FNU2)*(1.-2.*FNU2))
C
      PRINT 600,A,E1,E2,B,FNU1,FNU2,VF,FMU(1),FMU(2),FL,RHO(1),RHO(2),
     *   FN,FLAMDA(1),FLAMDA(2),GAMMA,TOL
  600 FORMAT(3H1A  ,E20.5,10X,6HE1     ,E17.5,10X,6HE2     ,E17.5,/,
     *         3H0B  ,E20.5,10X,6HNU1    ,E17.5,10X,6HNU2    ,E17.5,/,
     *         3H0VF ,E20.5,10X,6HMU 1   ,E17.5,10X,6HMU 2   ,E17.5,/,
     *         3H0L  ,E20.5,10X,6HRHO 1  ,E17.5,10X,6HRHO 2  ,E17.5,/,
     *         3H0N  ,E20.5,10X,6HLAMDA1,E17.5,10X,6HLAMDA2,E17.5,/,
     *      6H0GAMMA,E17.5, / ,6H0TOL   ,E17.5, / ,4H3   J,7X,6HDETERM,13X,
     *      1HR,17X,3HH 1,17X,3HH 2,17X,3HK 1,17X,3HK 2, /)
C
C          BEGIN ITERATIONS
C
    2 IFLAG = 0      $      PSQ = P*P        $      J = J+1
C
C          CALCULATE H 1  AND  H 2
C
      DO 10 I=1,2
      IMAG(I) = 3H          $       IH(I) = 0
      VARSQ = PSQ*RHO(I)/ (FLAMDA(I)+2.*FMU(I)) - GAMMASQ
      IF(VARSQ) 4, 3, 5
    3 IFLAG = 1       $       FH(I) = 0.       $       GO TO 10
    4 VARSQ = -VARSQ      $       IMAG(I) = 3H  I      $      IH(I) = 1
    5 FH(I) = SQRTF(VARSQ)
   10 CONTINUE
C
C          CALCULATE   K 1  AND  K 2
C
      DO 15 I=1,2
      IMAG(I+2) = 3H
      VARSQ = PSQ*RHO(I)/FMU(I) - GAMMASQ
      IF(VARSQ) 12, 11, 13
   11 IFLAG = 1       $       FK(I) = 0.       $       GO TO 15
   12 VARSQ = -VARSQ      $       IMAG(2*I+1) = 3H  I      $      IFLAG = 1
   13 FK(I) = SQRTF(VARSQ)
   15 CONTINUE
      IF(IFLAG) GO TO 40
C
```

```
C        GENERATE MATRIX
C
      R = A
      DO 30 I=1,3
      IND = (I+2)/2      $      ICOL = 2*I-1      $      IBES = (I+1)/2
C
      CALL BESLF(2, IBES, IND)
         AA(1,ICOL) = SIGN(I) * F1(IND)
         AA(2,ICOL) = SIGN(I) * F3(IND)
         AA(3,ICOL) = SIGN(I) * F5(IND)
         AA(4,ICOL) = SIGN(I) * F7(IND)
C
      ICOL = ICOL+1
      X = FK(IND) * A
      CALL BESSFL(X,3, BESL, IBES, 0)
         AA(1,ICOL) = SIGN(I) * F2(IND)
         AA(2,ICOL) = SIGN(I) * F4(IND)
         AA(3,ICOL) = SIGN(I) * F6(IND)
         AA(4,ICOL) = SIGN(I) * F8(IND)
   30 CONTINUE
C
         AA(5,1)  = AA(5,2)  =  AA(6,1)  =  AA(6,2)  =  0.
      R = B
      CALL BESLF(1, 1, 2)
         AA(5,3) = F5(2)
         AA(6,3) = F3(2) / FMU(2)
      CALL BESLF(1, 2, 2)
         AA(5,5) = F5(2)
         AA(6,5) = F3(2) / FMU(2)
C
      X = FK(2) * B
      CALL BESSEL(X, 3, BESL, 1, 0)
         AA(5,4) = F6(2)
         AA(6,4) = F4(2) / FMU(2)
      CALL BESSFL(X, 3, BESL, 2, 0)
         AA(5,6) = F6(2)
         AA(6,6) = F4(2) / FMU(2)
C
      CALL NWMATINV(AA,BB,CC,6,6,0,DET,IDET)
      JDET = 20 * IDET      $      DETRM = DET * (10.0**JDET)
C
      PRINT 601, J, DETRM, P, (FH(I), IMAG(I), I=1,4)
  601 FORMAT(I4, 2F17.8, 4( F17.7, A3), / )
C
      CALL ROOT(P, DETRM, TOL, DOMEGA, DIFF, II, JFLAG)
      IF(II) 35, 50,51
   35 PRINT 602      $      GO TO 1
  602 FORMAT(////, 10X, 20HITERATIONS DIVERGING)
C
   40 PRINT 603, J, P, (FH(I), IMAG(I), I=1,4)
  603 FORMAT(I4, 17X, E17.5, 4(E17.5, A3), /)
      JFLAG = 1
C
   50 IF(J .LT. ITER) GO TO 2
   51 FREQ = P / (2.*PI)
      PRINT 604, FREQ, DIFF
  604 FORMAT(//, 7H FREQ =,E17.8, 10X,7H DIFF =, E17.8)
      GO TO 1
  100 END
```

```
      FUNCTION COEF(I)
      COMMON/BESSLL/BESL(4)
      COMMON/DATA/FH(2), FK(2), FLAMDA(2), FMU(2), IH(2), R, PI, GAMMA,
     *   GAMMASQ
      DIMENSION SIGN(2)
      DATA(SIGN = 1., -1.)
C ***
      ENTRY F1
C
      J = IH(I) + 1
      HSQ = FH(I)**2
      COEF = ((FLAMDA(I)/2. + FMU(I))* SIGN(J)*HSQ * BESL(3)
     *        - FLAMDA(I)/R   * FH(I) * BESL(2)
     *      -((FLAMDA(I)/2. + FMU(I))* SIGN(J)*HSQ
     *      +  FLAMDA(I) * GAMMASQ) * BESL(1)
      RETURN
C ***
      ENTRY F2
C
      COEF = FMU(I) * GAMMA* FK(I)*(BESL(1) - BESL(3))
      RETURN
C ***
      ENTRY F3
      COEF = -2. * FMU(I) * GAMMA * FH(I) * BESL(2)
      RETURN
C ***
      ENTRY F4
C
      FKSQ = FK(I)**2
      COEF = FMU(I) * (FKSQ/4. * BESL(4)
     *                    -FK(I)/(2.*R) * BESL(3)
     *       -(1./(R*R) + .75*FKSQ - GAMMASQ) * BESL(2)
     *                    +FK(I)/(2.*R) * BESL(1))
      RETURN
C ***
      ENTRY F5
      COEF = -FH(I) * BESL(2)
      RETURN
C ***
      ENTRY F6
C
      COEF = GAMMA * BESL(2)
      RETURN
C ***
      ENTRY F7
C
      COEF = GAMMA * BESL(1)
      RETURN
C ***
      ENTRY F8
C
      COEF = -FK(I)/2. * BESL(3) + 1./R * BESL(2) + FK(I)/2. * BESL(1)
      RETURN     $     END
```

```
      SUBROUTINE BESLF(N,M,IND)
      COMMON/BESSLL/BESL(4)
      COMMON/DATA/FH(2), FK(2), FLAMDA(2), FMU(2), IH(2), R, PI, GAMMA,
     *   GAMMASQ
      X = FH(IND) * R
      IF(IH(IND)) GO TO 10
      CALL BESSEL (X, N, BESL, M, 0)
      RETURN
C
   10 IF(M .EQ. 2) GO TO 20
      CALL BESSEL (X, N, BESL, 3, 0)
C     BESL(1) =  BESL(1)
      BESL(2) = -BESL(2)
      BESL(3) = -BESL(3)
      RETURN
C
   20 CALL BESSEL(X, N, BESL, 4, 0)
      TOVRPI = 2./PI
      BESL(1) = -TOVRPI * BESL(1)
      BESL(2) = -TOVRPI * BESL(2)
      BESL(3) =  TOVRPI * BESL(3)
      RETURN     $     END
```

```
      SUBROUTINE NWMATINV(A,B,INDEX,NMAX,N,M,DETERM,IDET)
      DIMENSION A(NMAX,1),B(NMAX,1),INDEX(1)
      EQUIVALENCE(IROW,JROW,IR),(ICOLUM,JCOLUM,IC)
      EQUIVALENCE(AMAX,T,SWAP,IAMAX),(PIVOT,TEMP,ITEMP)
      DATA (MINUS=6000000000000000B)
C**   INITIALIZATION.
      DETERM=1.0
      IDET = 0
      DO 20 J=1,N
   20 INDEX(J)=MINUS
      DO 550 I=1,N
C**   SEARCH FOR ELEMENT OF LARGEST MAGNITUDE.
      AMAX=0.0
      DO 105 J1=1,N
      IF(-INDEX(J1)) 105,105,60
   60 DO 100 K1=1,N
      IF(-INDEX(K1)) 100,100,80
   80 TEMP=A(K1,J1)
      IF(TEMP)83,100,82
   82 TEMP=-TEMP
   83 IF(-ITEMP-IAMAX) 100,100,84
   84 AMAX=-TEMP
      IR=K1
      IC=J1
  100 CONTINUE
  105 CONTINUE
      IF(AMAX)120,115
  115 DETERM=0
      IDET = 0
      RETURN
  120 IROW=IR
      ICOLUM=IC
      INDEX(ICOLUM)=INDEX(ICOLUM).AND..NOT.MINUS
      IF(.NOT.(IROW-ICOLUM))260,140
  140 DETERM=-DETERM
C**   EXCHANGE ROWS.
      DO 200 L=1,N
      SWAP=A(IROW,L)
      A(IROW,L)=A(ICOLUM,L)
  200 A(ICOLUM,L)=SWAP
      IF(.NOT.M)260,210
  210 DO 250 L=1, M
      SWAP=B(IROW,L)
      B(IROW,L)=B(ICOLUM,L)
  250 B(ICOLUM,L)=SWAP
C**   SAVE PIVOT INFORMATION. DO DETERMINANT.
  260 INDEX(I)=IROW*1000000000B+ICOLUM+INDEX(I)
      PIVOT   =A(ICOLUM,ICOLUM)
      DETERM=DETERM*PIVOT
C
C  **     KEEP DETERMINANT BETWEEN 1.0E-20 AND 1.0E+20
  270 DETERM1 = ABSF(DETERM)
      IF(DETERM1 .GT. 1.E-20) GO TO 275
      DETERM = DETERM * 1.E20
      IDET = IDET - 20
      GO TO 270
  275 IF(DETERM1 .LT. 1.E20) GO TO 300
      DETERM = DETERM / 1.E20
      IDET = IDET + 20
      GO TO 270
  300 CONTINUE
C**   REDUCE LEADING COEF TO 1.
```

```
      A(ICOLUM,ICOLUM)=1.0
      DO 350 L=1,N
  350 A(ICOLUM,L)=A(ICOLUM,L)/PIVOT
      IF(.NOT.M)380,360
  360 DO 370 L=1,M
  370 B(ICOLUM,L)=B(ICOLUM,L)/PIVOT
C**   SUBSTITUTE FOR NTH VARIABLE.
  380 DO 550 L1=1,N
      IF(.NOT.(L1-ICOLUM)) 550,400
  400 T=A(L1,ICOLUM)
      A(L1,ICOLUM)=0.0
      DO 450 L=1,N
  450 A(L1,L)=A(L1,L)-A(ICOLUM,L)*T
      IF(.NOT.M)550,460
  460 DO 500 L=1,M
  500 B(L1,L)=B(L1,L)-B(ICOLUM,L)*T
  550 CONTINUE
C**   UNDO ROW EXCHANGES.
      L=N
      DO 710 L2=1,N
      JROW=INDEX(L)/100000000B
      JROW=JROW.AND.77777B
      JCOLUM=INDEX(L).AND.77777B
      IF(.NOT.(JROW-JCOLUM))710,630
  630 DO 705 K=1,N
      SWAP=A(K,JROW)
      A(K,JROW)=A(K,JCOLUM)
  705 A(K,JCOLUM)=SWAP
  710 L=L-1
  740 RETURN
      END
```

```
      SUBROUTINE ROOT(X, Y, TOL, DEL, DIFF, IFLAG, JFLAG)
C
C        THIS SUBROUTINE WILL FIND A ROOT BY FALSE POSITION
C
      IFLAG = 0
      GO TO (10, 20, 30) JFLAG
C
   10 X1 = X         $      Y1 = Y
      JFLAG = 2
   15 X = X + DEL    $      RETURN
C
   20 X2 = X         $      Y2 = Y
      IF (Y1 * Y2) 25, 30, 21
   21 X1 = X2        $      Y1 = Y2        $      GO TO 15
   25 JFLAG = 3
   26 X = X2 - ((X2-X1)/(Y2-Y1)) * Y2  $   RETURN
C
   30 X3 = X         $      Y3 = Y
      DIFF = ABSF(X3 - X2)
      IF(DIFF .LT. TOL) GO TO 50
   31 IF(Y1 * Y3) 32, 50, 33
   32 X2 = X3        $      Y2 = Y3        $      GO TO 26
   33 IF(Y2 * Y3) 34, 50, 40
   34 X1 = X3        $      Y1 = Y3        $      GO TO 26
C
   40 IFLAG = -1     $      RETURN
C
   50 IFLAG = 1      $      RETURN         $      END
```

```
      PROGRAM DETERM
C
C        CIRCULAR OUTER BOUNDARY
C
      COMMON/INPUT/A,B,G1,G2,FLAMDA1,FLAMDA2,BETA,BETASQ
      COMMON/BESSL/BESSELJ(2,2,3), BESSELY(2,2,3), BESSELI(2,2,3)
      COMMON/MU/FMU1AB, FMU2AB, FMU1GD, FMU2GD
      DIMENSION C(4,4), SCRATCH(6), FMU(4), IMAG(4)
      DIMENSION II(4)
      EQUIVALENCE(FMU(1), FMU1AB)
      DATA(PI = 3.141592653)
C
    1 READ 700, VF, FN, OMEGA, DOMEGA, TOL, ITER
  700 FORMAT(5E15.5, I5)
C
      IF(EOF, 50) 100, 10
   10 A = 2.5E-3
      E1 = 1.0E7
      FNU1 = 2.0E-1
      E2 = 3.6E5      $     FNU2 = 3.5E-1
      RHO1 = .0938/(32.2*12.)      $       RHO2 = .0448/(32.2*12.)
      FL = 3.
      J = 0      $       JFLAG = 1
      B = A * SQRTF(1./VF)
      FLAMDA1 = E1*FNU1/((1.+FNU1)*(1.-2.*FNU1))
      FLAMDA2 = E2*FNU2/((1.+FNU2)*(1.-2.*FNU2))
      G1 = E1/(2.*(1.+FNU1))      $       G2 = E2/(2.*(1.+FNU2))
      C I 1SQ =(2.*G1+FLAMDA1)/RHO1
      C I 2SQ =(2.*G2+FLAMDA2)/RHO2
      C II1SQ = G1/RHO1
      C II2SQ = G2/RHO2
      BETA = FN*PI/FL
      BETASQ = BETA**2
      PRINT 600, A, E1, E2, B, FNU1, FNU2, VF, G1, G2, FL, RHO1, RHO2,
     *   FN, FLAMDA1, FLAMDA2, BETA, TOL
  600 FORMAT(3H1A  ,E20.5,10X,6HE1    ,E17.5,10X,6HE2      ,E17.5,/,
     *        3H0B  ,E20.5,10X,6HNU1    ,E17.5,10X,6HNU2     ,E17.5,/,
     *        3H0VF ,E20.5,10X,6HG  1   ,E17.5,10X,6HG  2    ,E17.5,/,
     *        3H0L  ,E20.5,10X,6HRHO 1  ,E17.5,10X,6HRHO 2   ,E17.5,/,
     *        3H0N  ,E20.5,10X,6HLAMDA1,E17.5,10X,6HLAMDA2,E17.5,/,
     *     6H0BETA  ,F17.5, / ,6H0TOL   ,F17.5, / ,4H3   J,7X,6HOMEGA ,11X,
     *     5HDETRM,10X,11HFMU1AB (H2),9X,11HFMU2AB (K2),9X,11HFMU1GD (H1)
     *          9X,11HFMU2GD (K1), /)
C
    2 OMEGASQ = OMEGA * OMEGA
      J = J+1
      FMU(1) = OMEGASQ/C I 2SQ - BETASQ
      FMU(2) = OMEGASQ/C II2SQ - BETASQ
      FMU(3) = OMEGASQ/C I 1SQ - BETASQ
      FMU(4) = OMEGASQ/C II1SQ - BETASQ
C
      K = 0
      DO 4 I=1,4      $       IMAG(I) = 3H      $       II(I) = 0
      IF(FMU(I)) 3, 4, 4
    3 K = K+1      $      IMAG(I) = 3H  I      $       II(K) = I
      FMU(I) = -FMU(I)
    4 CONTINUE
C
      FMU1AB = SQRTF( FMU(1) )
      FMU2AB = SQRTF( FMU(2) )
      FMU1GD = SQRTF( FMU(3) )
      FMU2GD = SQRTF( FMU(4) )
```

```
      IF(K-1) 75, 5, 75
    5 IF(II(1) - 3) 75, 6, 75
C
    6 CALL BESLFUN
C
   11 C(1,1) = FMU1AB*BESSELJ(2,1,2)
   12 C(1,2) = FMU1AB*BESSELY(2,1,2)
   13 C(1,3) = -BETA*BESSELJ(2,2,2)
   14 C(1,4) = -BETA*BESSELY(2,2,2)
   15 C(1,5) = C(1,6) = C(2,5) = C(2,6) = 0.
C
   21 C(2,1) = 2.*FMU1AB*BETA*BESSELJ(2,1,2)
   22 C(2,2) = 2.*FMU1AB*BETA*BESSELY(2,1,2)
   23 C(2,3) = (FMU2AB**2-BETASQ)*BESSELJ(2,2,2)
   24 C(2,4) = (FMU2AB**2-BETASQ)*BESSELY(2,2,2)
C
   31 C(3,1) = FMU1AB*BESSELJ(2,1,1)
   32 C(3,2) = FMU1AB*BESSELY(2,1,1)
   33 C(3,3) = -BETA*BESSELJ(2,2,1)
   34 C(3,4) = -BETA*BESSELY(2,2,1)
   35 C(3,5) = FMU1GD*BESSELI(2,1,3)
   36 C(3,6) = BETA*BESSELJ(2,2,3)
C
   41 C(4,1) = BETA*BESSELJ(1,1,1)
   42 C(4,2) = BETA*BESSELY(1,1,1)
   43 C(4,3) = FMU2AB*BESSELJ(1,2,1)
   44 C(4,4) = FMU2AB*BESSELY(1,2,1)
   45 C(4,5) = -BETA*BESSELI(1,1,3)
   46 C(4,6) = -FMU2GD*BESSELJ(1,2,3)
C
   51 C(5,1) = (-FLAMDA2*BETASQ-(FLAMDA2+2.*G2)*FMU1AB**2)*
     *        BESSELJ(1,1,1)+2.*G2*FMU1AB*BESSELJ(2,1,1))/A
   52 C(5,2) = (-FLAMDA2*BETASQ-(FLAMDA2+2.*G2)*FMU1AB**2)*
     *        BESSELY(1,1,1)+2.*G2*FMU1AB*BESSELY(2,1,1))/A
   53 C(5,3) = -2.*G2*FMU2AB*BETA*(BESSELJ(1,2,1)-BESSELJ(2,2,1)/
     *        (FMU2AB*A))
   54 C(5,4) = -2.*G2*FMU2AB*BETA*(BESSELY(1,2,1)-BESSELY(2,2,1)/
     *        (FMU2AB*A))
   55 C(5,5) =-(FLAMDA1*BETASQ-(FLAMDA1+2.*G1)*FMU1GD**2)*BESSELI(1,1,3)
     *        -2.*G1*FMU1GD*BESSELI(2,1,3))/A
   56 C(5,6) = 2.*G1*FMU2GD*BETA*(BESSELJ(1,2,3)-BESSELJ(2,2,3)/
     *        (FMU2GD*A))
   57 C(5,1) = -C(5,1)         $        C(5,2) = -C(5,2)
C
   61 C(6,1) = 2.*G2*FMU1AB*BETA*BESSELJ(2,1,1)
   62 C(6,2) = 2.*G2*FMU1AB*BETA*BESSELY(2,1,1)
   63 C(6,3) = G2*(FMU2AB**2-BETASQ)*BESSELJ(2,2,1)
   64 C(6,4) = G2*(FMU2AB**2-BETASQ)*BESSELY(2,2,1)
   65 C(6,5) = 2.*G1*FMU1GD*BETA*BESSELI(2,1,3)
   66 C(6,6) = -G1*(FMU2GD**2-BETASQ)*BESSELJ(2,2,3)
C
      CALL NWMATINV(C, BB, SCRATCH, 6, 6, 0, DET, IDET)
      JDET = 20*IDET       $      DETRM = DET * (10.**JDET)
C
      PRINT 601, J, OMEGA, DETRM, (FMU(I), IMAG(I), I=1,4)
  601 FORMAT(I4, 2E17.8, 4( F17.7, A3), / )
C
      CALL ROOT(OMEGA, DETRM, TOL, DOMEGA, DIFF, JJ, JFLAG)
      IF(JJ) 70, 80, 85
   70 PRINT 602       $       GO TO 1
  602 FORMAT(////, 10X, 20HITERATIONS DIVERGING)
```

```
C
   75 PRINT 603, J, OMEGA, (FMU(I), IMAG(I), I=1,4)
  603 FORMAT(I4, E17.5, 17X, 4(E17.5, A3), /)
      JFLAG = 1
C
   80 IF(J .LT. ITER) GO TO 2
   85 FREQ = OMEGA/(2.*PI)
      PRINT 604, FREQ, DIFF
  604 FORMAT(//, 7H FREQ =,E17.8, 10X,7H DIFF =, E17.8)
      GO TO 1
  100 END
```

```
      SUBROUTINE BESLFUN
      COMMON/INPUT/A,B,G1,G2,FLAMDA1,FLAMDA2,BETA,BETASQ
      COMMON/BESSL/BESSELJ(2,2,3), BESSELY(2,2,3), BESSELI(2,2,3)
      COMMON/MU/FMU1AB, FMU2AB, FMU1GD, FMU2GD
      DIMENSION X(3), XX(3), ANS(2), ANSW(2)
      X(1) = FMU1AB*A        $      XX(1) = FMU2AB*A
      X(2) = FMU1AB*B        $      XX(2) = FMU2AB*B
      X(3) = FMU1GD*A        $      XX(3) = FMU2GD*A
C
      DO 10 I=1,3
      CALL BESSEL(X(I), 1, ANS, 1, 0)
      CALL BESSEL(XX(I),1, ANSW,1, 0)
      BESSELJ(1,1,I) = ANS(1)        $      BESSELJ(2,1,I) = ANS(2)
      BESSELJ(1,2,I) = ANSW(1)       $      BESSELJ(2,2,I) = ANSW(2)
      CALL BESSEL(X(I), 1, ANS, 2, 0)
      CALL BESSEL(XX(I),1, ANSW,2, 0)
      BESSELY(1,2,I) = ANSW(1)       $      BESSELY(2,2,I) = ANSW(2)
      BESSELY(1,1,I) = ANS(1)        $      BESSELY(2,1,I) = ANS(2)
   10 CONTINUE
C
      CALL BESSEL(X(3), 1, ANS, 3, 0)
      BESSELI(1, 1, 3) = ANS(1)      $      BESSELI(2,1,3) = ANS(2)
C
      RETURN     $     END
```

```
      SUBROUTINE NWMATINV(A,B,INDEX,NMAX,N,M,DETERM,IDET)               NWMV   1
      DIMENSION A(NMAX,1),B(NMAX,1),INDEX(1)                            NWMV   2
      EQUIVALENCE(IROW,JROW,IR),(ICOLUM,JCOLUM,IC)                      NWMV   3
      EQUIVALENCE(AMAX,T,SWAP,IAMAX),(PIVOT,TEMP,ITEMP)                 NWMV   4
      DATA (MINUS=6000000000000000B)                                    NWMV   5
C**   INITIALIZATION.
      DETERM=1.0                                                        NWMV   6
      IDET = 0
      DO 20 J=1,N                                                       NWMV   7
   20 INDEX(J)=MINUS                                                    NWMV   8
      DO 550 I=1,N                                                      NWMV   9
C**   SEARCH FOR ELEMENT OF LARGEST MAGNITUDE.
      AMAX=0.0                                                          NWMV  10
      DO 105 J1=1,N                                                     NWMV  11
      IF(-INDEX(J1)) 105,105,60                                         NWMV  12
   60 DO 100 K1=1,N                                                     NWMV  13
      IF(-INDEX(K1)) 100,100,80                                         NWMV  14
   80 TEMP=A(K1,J1)                                                     NWMV  15
      IF(TEMP)83,100,82                                                 NWMV  16
   82 TEMP=-TEMP                                                        NWMV  17
   83 IF(-ITEMP-IAMAX) 100,100,84                                       NWMV  18
   84 AMAX=-TEMP                                                        NWMV  19
      IR=K1                                                             NWMV  20
      IC=J1                                                             NWMV  21
  100 CONTINUE                                                          NWMV  22
  105 CONTINUE                                                          NWMV  23
      IF(AMAX)120,115                                                   NWMV  24
  115 DETERM=0                                                          NWMV  25
      IDET = 0
      RETURN                                                            NWMV  26
  120 IROW=IR                                                           NWMV  27
      ICOLUM=IC                                                         NWMV  28
      INDEX(ICOLUM)=INDEX(ICOLUM).AND..NOT.MINUS                        NWMV  29
      IF(.NOT.(IROW-ICOLUM))260,140                                     NWMV  30
  140 DETERM=-DETERM                                                    NWMV  31
C**   EXCHANGE ROWS.
      DO 200 L=1,N                                                      NWMV  32
      SWAP=A(IROW,L)                                                    NWMV  33
      A(IROW,L)=A(ICOLUM,L)                                             NWMV  34
  200 A(ICOLUM,L)=SWAP                                                  NWMV  35
      IF(.NOT.M)260,210                                                 NWMV  36
  210 DO 250 L=1, M                                                     NWMV  37
      SWAP=B(IROW,L)                                                    NWMV  38
      B(IROW,L)=B(ICOLUM,L)                                             NWMV  39
  250 B(ICOLUM,L)=SWAP                                                  NWMV  40
C**   SAVE PIVOT INFORMATION, DO DETERMINANT.
  260 INDEX(I)=IROW*1000000000B+ICOLUM+INDEX(I)                         NWMV  41
      PIVOT    =A(ICOLUM,ICOLUM)                                        NWMV  42
      DETERM=DETERM*PIVOT                                               NWMV  43
C
C  C*    KEEP DETERMINANT BETWEEN 1.0E-20 AND 1.0E+20
  270 DETERM1 = ABSF(DETERM)
      IF(DETERM1 .GT. 1.E-20) GO TO 275
      DETERM = DETERM * 1.E20
      IDET = IDET - 20
      GO TO 270
  275 IF(DETERM1 .LT. 1.E20) GO TO 300
      DETERM = DETERM / 1.E20
      IDET = IDET + 20
      GO TO 270
  300 CONTINUE
C**   REDUCE LEADING COEF. TO 1.
```

```
      A(ICOLUM,ICOLUM)=1.0
      DO 350 L=1,N
  350 A(ICOLUM,L)=A(ICOLUM,L)/PIVOT
      IF(.NOT.M)380,360
  360 DO 370 L=1,M
  370 B(ICOLUM,L)=B(ICOLUM,L)/PIVOT
C**   SUBSTITUTE FOR NTH VARIABLE.
  380 DO 550 L1=1,N
      IF(.NOT.(L1-ICOLUM)) 550,400
  400 T=A(L1,ICOLUM)
      A(L1,ICOLUM)=0.0
      DO 450 L=1,N
  450 A(L1,L)=A(L1,L)-A(ICOLUM,L)*T
      IF(.NOT.M)550,460
  460 DO 500 L=1,M
  500 B(L1,L)=B(L1,L)-B(ICOLUM,L)*T
  550 CONTINUE
C**   UNDO ROW EXCHANGES.
      L=N
      DO 710 L2=1,N
      JROW=INDEX(L)/1000000000B
      JROW=JROW.AND.77777B
      JCOLUM=INDEX(L).AND.77777B
      IF(.NOT.(JROW-JCOLUM))710,630
  630 DO 705 K=1,N
      SWAP=A(K,JROW)
      A(K,JROW)=A(K,JCOLUM)
  705 A(K,JCOLUM)=SWAP
  710 L=L-1
  740 RETURN
      END
```

```
      SUBROUTINE ROOT(X, Y, TOL, DEL, DIFF, IFLAG, JFLAG)
C
C        THIS SUBROUTINE WILL FIND A ROOT BY FALSE POSITION
C
      IFLAG = 0
      GO TO (10, 20, 30) JFLAG
C
   10 X1 = X          $     Y1 = Y
      JFLAG = 2
   15 X = X + DEL     $     RETURN
C
   20 X2 = X          $     Y2 = Y
      IF (Y1 * Y2) 25, 50, 21
   21 X1 = X2         $     Y1 = Y2         $     GO TO 15
   25 JFLAG = 3
   26 X = X2 - ((X2-X1)/(Y2-Y1)) * Y2       $     RETURN
C
   30 X3 = X          $     Y3 = Y
      DIFF = ABSF(X3 - X2)
      IF(DIFF .LT. TOL) GO TO 50
   31 IF(Y1 * Y3) 32, 50, 33
   32 X2 = X3         $     Y2 = Y3         $     GO TO 26
   33 IF(Y2 * Y3) 34, 50, 40
   34 X1 = X3         $     Y1 = Y3         $     GO TO 26
C
   40 IFLAG = -1      $     RETURN
C
   50 IFLAG = 1       $     RETURN          $     END
```

APPENDIX X

PROGRAM FOR FORCED VIBRATION USING SIMPLIFIED ANALYSIS

```
100 $USE LINEQ$***
110' *** PROGRAM FOR FORCED VIBRATION USING SIMPLIFIED ANALYSIS
120' *** WRITTEN JUNE 18, 1968 BY GEORGE BURGIN ***
125' *** FREE-FREE CASE WITH U R (B) = ZERO.
130 DIMENSION AA(25,25),RHS(25)
140 105 CONTINUE
150 A = 0.0025
160 VF = 0.65
170 RHO1=2.42734E-4
180 RHO2=1.15942E-4
190 E1=EF=10000000.
200 E2=ER=360000.
210 FNU1=FNUF=0.2
220 FNU2=FNUR=0.35
230 G1=GF=E1/(2.+2.*FNU1)
240 G2=GR=E2/(2.+2.*FNU2)
250 PI=3.141592653
260 192 FORMAT("A = ",E10.4,"   B =",E10.4,"   VF =",E10.4/
270 +"RHO1 =",E10.4,"   RHO2 =",E10.4,"   E 1 =",E10.4,/
280 +"E 2 = ",E10.4,"   NU 1 = ",E10.4,"   NU 2 = ",E10.4,/
290 +"G 1 =",E10.4,"   G 2 = ",E10.4,"   FL = ",E10.4///)
300 B=A*SQRTF(1./VF)
310 FL=3.
320 PRINT 191
330 191 FORMAT(///"INPUT DATA"/"----------"//)
340 PRINT 192,A,B,VF,RHO1,RHO2,E1,E2,FNU1,FNU2,G1,G2,FL
350 AA(1,1)=A
360 AA(1,2)=-A
370 AA(1,3)=-1./A
380 AA(2,1)=EF/((1.+FNUF)*(1.-2.*FNUF))
390 AA(2,2)=-ER/((1.+FNUR)*(1.-2.*FNUR))
400 AA(2,3)=ER/((1.+FNUR)*A**2)
410 AA(3,1) = 0.0
420 AA(3,2)=B
430 AA(3,3) = 1./B
440 RHS(1)=A*(FNU2-FNU1)
445 RHS(1)=-RHS(1)
450 RHS(2) =0.0
460 RHS(3)=B*FNU2
470 CALL LINEQ(AA,RHS,3,1)
480 FK1F=RHS(1)
490 FK1R=RHS(2)
500 FK2R=RHS(3)
510 FLANDAR=FNUR*ER/((1.+FNUR)*(1.-2.*FNUR))
520 FLANDAF=FNUF*EF/((1.+FNUF)*(1.-2.*FNUF))
530 T1=RHO1*PI*(-FNU1+FK1F)**2*A**4/4.
540 T2=(-FNU2+FK1R)**2*(B**4-A**4)/4.
550 T3=(-FNU2+FK1R)*FK2R*(B**2-A**2)
560 T4=FK2R**2*LOGF(B/A)
570 C1=T1+RHO2*PI*(T2+T3+T4)
```

FORCE4 CONT…D

```
580 C2=RHO1*PI*A**2/2.+RHO2*PI*(B*B-A*A)/2.
590 T5=EF/2.*A**2
600 T6=ER*(B**2-A**2)/2.
610 T7=EF/((1.+FNUF)*(1.-2.*FNUF))*FK1F**2*A**2
620 T8=ER/((1.+FNUR)*(1.-2.*FNUR))*FK1R**2*(B**2-A**2)
630 T9=-ER/(1.+FNUR)*FK2R**2*(1./B**2-1./A**2)
650 C3=PI*(T5+T6+T7+T8+T9)
660 PRINT 194,FK1F,FK1R,FK2R
670 194 FORMAT("K 1 F =",E10.4,"   K 1 R =",E10.4,"   K 2 R =",E10.4/)
680 PRINT 195,C1,C2,C3
690 195 FORMAT("OMEGA1 =",E10.4,"   OMEGA2 =",E10.4,"  OMEGA 3 =",E10.4)
694 PRINT,""
695 DO 726 N=1,5
696 726 CALL OMEGANAT(C1,C2,C3,N,FL)
700 106 CONTINUE; PRINT 92
710 92 FORMAT (///"INPUT VALUE FOR OMEGA E "); INPUT,OMEGAE
720 OMEGA1=C1;OMEGA2=C2;OMEGA3=C3
730 BETA1=OMEGA3-OMEGAE**2*OMEGA1
740 BETA2=OMEGA2*OMEGAE**2
750 F11=SQRT(BETA1/BETA2);SQ1=SQRT(BETA2/BETA1)
755 PRINT 291,SQ1
756 291 FORMAT (//" BETA = ",E12.5)
760 F12=F11/SIN(SQ1*FL)
770' *** START DO-LOOP ON Z *********
780 DELTAZ=1.
782 Z=0.0
785 Z=0.0
790 DO 999 KOUNT=1,3
800 Z= Z+DELTAZ
805 PRINT 97,Z
810 ARG=SQRT(OMEGA2*OMEGAE**2/(OMEGA3-OMEGAE**2*OMEGA1))
820 DPHIDZ=F12*ARG*SIN(ARG*Z)
830 EPSZ=DPHIDZ
840 PRINT 94,EPSZ
850 94 FORMAT(/"EPS Z = ",E12.4/)
860 GO TO 1575
880 880 CONTINUE
882 FK1F=C1SUP1/EPSZ
883 FK1R=C1SUP2/EPSZ
884 FK2R=C2SUP2/EPSZ
900 97 FORMAT(//"---------------------------------------------------------
910 +----------"//20X," Z = ",F12.2/)
920 PRINT 55,C1SUP1,C1SUP2,C2SUP2
930 55 FORMAT(//"C 1 SUP 1 =",1PE20.7/"C 1 SUP 2 =",E20.7/
940 +"C 2 SUP 2 =",E20.7)
950 SIGMA1=E1/((1.+FNU1)*(1.-2.*FNU1))*C1SUP1
960 SIGMA1Z=EPSZ*E1+2.*FNU1*SIGMA1
970 PRINT 57,SIGMA1,SIGMA1Z
789 57 FORMAT(//"SIGMA 1 R = SIGMA 1 THETA =",1PE9.3,"   SIGMA Z =",
981 +1PE9.3)
```

FORCE4 CONTINUED

```
UR1=(-FNU1+FK1F)*EPSZ*A
W=-F11*COS(SQ1*Z)/SIN(SQ1*FL)
PRINT 995,UR1,W
995 FORMAT (" U R 1 =",1PE15.5,"   W 1 =",E15.4)
DELTAR=(B-A)/4.;R=A
DO 95 KOUNTR=1,5
FMULT=E2/((1.+FNU2)*(1.-2.*FNU2))
SIGMAR=FMULT*(C1SUP2-(1.-2.*FNU2)* 2SUP2/R**2)
SIGMAT=FMULT*(C1SUP2+(1.-2.*FNU2)*C2SUP2/R**2)
SIGMAZ=EPSZ*E2+FNU2*(SIGMAR + SIGMAT)
PRINT 56,Z,R,SIGMAR,SIGMAT,SIGMAZ
UR2=(-FNU2+FK1R)*EPSZ*R + FK2R*EPSZ/R
PRINT 393,UR2
393 FORMAT( " U R 2 =",1PE15.5)
56 FORMAT(///15X,"Z = "F8.2,"   R = ",F11.7/
+"SIGMA R=",1PE9.3,"   SIGMA THETA =",1PE9.3,"   SIGMAZ =",1PE9.3)
R = R + DELTAR
95 CONTINUE
999 CONTINUE
STO[redacted]
1575 CONTINUE
AA(1,1)=A
AA(1,2)=-A
AA(1,3)=-1./A
AA(2,1)=EF/((1.+FNUF)*(1.-2.*FNUF))
AA(2,2)=-ER/((1.+FNUR)*(1.-2.*FNUR))
AA(2,3)=ER/((1.+FNUR)*A**2)
AA(3,1) = 0.0
AA(3,2)=B
AA(3,3) = 1./B
RHS(1)=-EPSZ*A*(FNU2-FNU1)
RHS(2) =0.0
RHS(3)=B*FNU2*EPSZ
CALL LINEQ(AA,RHS,3,1)
C1SUP1=RHS(1)
C1SUP2=RHS(2)
C2SUP2=RHS(3)
GO TO 880
SUBROUTINE OMEGANAT(C1,C2,C3,N,FL)
PI = 4.*ATAN(1.)
FN = N
RAD=C3/(C2+FN**2*PI**2/FL**2*C1)
OMEGA=FN*PI/FL*SQRT(RAD)
PRINT 91,N,OMEGA
91 FORMAT("OMEGA NATURAL SUB",I3, " = ",1PE10.4)
RETURN;END
```

APPENDIX XI

COMPOSITE VELOCITIES

VF = 5.00000E-01
E1 = 1.00000E+07
RHO1 = 2.42800E-04
NU1 = 2.00000E-01

RO1/RO2	NU1/NU2	E1/E2	C
1.50	0.66667	50.0	1.60375E+05
1.50	0.66667	100.0	1.58812E+05
1.50	0.66667	150.0	1.58280E+05
1.50	0.66667	200.0	1.58012E+05
1.50	0.57143	50.0	1.61140E+05
1.50	0.57143	100.0	1.59210E+05
1.50	0.57143	150.0	1.58549E+05
1.50	0.57143	200.0	1.58215E+05
2.00	0.66667	50.0	1.69050E+05
2.00	0.66667	100.0	1.67403E+05
2.00	0.66667	150.0	1.66842E+05
2.00	0.66667	200.0	1.66559E+05
2.00	0.57143	50.0	1.69856E+05
2.00	0.57143	100.0	1.67822E+05
2.00	0.57143	150.0	1.67125E+05
2.00	0.57143	200.0	1.66774E+05
2.50	0.66667	50.0	1.74983E+05
2.50	0.66667	100.0	1.73278E+05
2.50	0.66667	150.0	1.72698E+05
2.50	0.66667	200.0	1.72405E+05
2.50	0.57143	50.0	1.75818E+05
2.50	0.57143	100.0	1.73712E+05
2.50	0.57143	150.0	1.72991E+05
2.50	0.57143	200.0	1.72627E+05

VF = 6.00000E-01
E1 = 1.00000E+07
RHO1 = 2.42800E-04
NU1 = 2.00000E-01

RO1/RO2	NU1/NU2	E1/E2	C
1.50	0.66667	50.0	1.71673E+05
1.50	0.66667	100.0	1.70298E+05
1.50	0.66667	150.0	1.69826E+05
1.50	0.66667	200.0	1.69587E+05
1.50	0.57143	50.0	1.72372E+05
1.50	0.57143	100.0	1.70666E+05
1.50	0.57143	150.0	1.70076E+05
1.50	0.57143	200.0	1.69776E+05
2.00	0.66667	50.0	1.78682E+05
2.00	0.66667	100.0	1.77251E+05
2.00	0.66667	150.0	1.76760E+05
2.00	0.66667	200.0	1.76512E+05
2.00	0.57143	50.0	1.79410E+05
2.00	0.57143	100.0	1.77635E+05
2.00	0.57143	150.0	1.77021E+05
2.00	0.57143	200.0	1.76709E+05
2.50	0.66667	50.0	1.83324E+05
2.50	0.66667	100.0	1.81856E+05
2.50	0.66667	150.0	1.81352E+05
2.50	0.66667	200.0	1.81097E+05
2.50	0.57143	50.0	1.84071E+05
2.50	0.57143	100.0	1.82250E+05
2.50	0.57143	150.0	1.81619E+05
2.50	0.57143	200.0	1.81300E+05

VF = 7.00000E-01
E1 = 1.00000E+07
RHO1 = 2.42800E-04
NU1 = 2.00000E-01

RO1/RO2	NU1/NU2	E1/E2	C
1.50	0.66667	50.0	1.81643E+05
1.50	0.66667	100.0	1.80360E+05
1.50	0.66667	150.0	1.79911E+05
1.50	0.66667	200.0	1.79683E+05
1.50	0.57143	50.0	1.82305E+05
1.50	0.57143	100.0	1.80718E+05
1.50	0.57143	150.0	1.80157E+05
1.50	0.57143	200.0	1.79869E+05
2.00	0.66667	50.0	1.86909E+05
2.00	0.66667	100.0	1.85589E+05
2.00	0.66667	150.0	1.85127E+05
2.00	0.66667	200.0	1.84892E+05
2.00	0.57143	50.0	1.87590E+05
2.00	0.57143	100.0	1.85957E+05
2.00	0.57143	150.0	1.85380E+05
2.00	0.57143	200.0	1.85084E+05
2.50	0.66667	50.0	1.90298E+05
2.50	0.66667	100.0	1.88953E+05
2.50	0.66667	150.0	1.88483E+05
2.50	0.66667	200.0	1.88244E+05
2.50	0.57143	50.0	1.90991E+05
2.50	0.57143	100.0	1.89328E+05
2.50	0.57143	150.0	1.88740E+05
2.50	0.57143	200.0	1.88439E+05

VF = 8.00000E-01
E1 = 1.00000E+07
RHO1 = 2.42800E-04
NU1 = 2.00000E-01

RO1/RO2	NU1/NU2	E1/E2	C
1.50	0.66667	50.0	1.90716E+05
1.50	0.66667	100.0	1.89396E+05
1.50	0.66667	150.0	1.88917E+05
1.50	0.66667	200.0	1.88669E+05
1.50	0.57143	50.0	1.91371E+05
1.50	0.57143	100.0	1.89769E+05
1.50	0.57143	150.0	1.89177E+05
1.50	0.57143	200.0	1.88869E+05
2.00	0.66667	50.0	1.94216E+05
2.00	0.66667	100.0	1.92872E+05
2.00	0.66667	150.0	1.92384E+05
2.00	0.66667	200.0	1.92131E+05
2.00	0.57143	50.0	1.94883E+05
2.00	0.57143	100.0	1.93251E+05
2.00	0.57143	150.0	1.92649E+05
2.00	0.57143	200.0	1.92335E+05
2.50	0.66667	50.0	1.96410E+05
2.50	0.66667	100.0	1.95051E+05
2.50	0.66667	150.0	1.94558E+05
2.50	0.66667	200.0	1.94302E+05
2.50	0.57143	50.0	1.97085E+05
2.50	0.57143	100.0	1.95435E+05
2.50	0.57143	150.0	1.94826E+05
2.50	0.57143	200.0	1.94508E+05

VF	=	5.00000E-01
E1	=	6.00000E+07
RHO1	=	2.42800E-04
NU1	=	2.00000E-01

RO1/RO2	NU1/NU2	E1/E2	C
1.50	0.66667	50.0	3.92837E+05
1.50	0.66667	100.0	3.89008E+05
1.50	0.66667	150.0	3.87706E+05
1.50	0.66667	200.0	3.87049E+05
1.50	0.57143	50.0	3.94710E+05
1.50	0.57143	100.0	3.89983E+05
1.50	0.57143	150.0	3.88364E+05
1.50	0.57143	200.0	3.87547E+05
2.00	0.66667	50.0	4.14086E+05
2.00	0.66667	100.0	4.10051E+05
2.00	0.66667	150.0	4.08678E+05
2.00	0.66667	200.0	4.07986E+05
2.00	0.57143	50.0	4.16060E+05
2.00	0.57143	100.0	4.11078E+05
2.00	0.57143	150.0	4.09372E+05
2.00	0.57143	200.0	4.08510E+05
2.50	0.66667	50.0	4.28620E+05
2.50	0.66667	100.0	4.24443E+05
2.50	0.66667	150.0	4.23022E+05
2.50	0.66667	200.0	4.22305E+05
2.50	0.57143	50.0	4.30664E+05
2.50	0.57143	100.0	4.25506E+05
2.50	0.57143	150.0	4.23740E+05
2.50	0.57143	200.0	4.22848E+05

VF = 6.00000E-01
E1 = 6.00000E+07
RHO1 = 2.42600E-04
NU1 = 2.00000E-01

RO1/RO2	NU1/NU2	E1/E2	C
1.50	0.66667	50.0	4.20510E+05
1.50	0.66667	100.0	4.17142E+05
1.50	0.66667	150.0	4.15986E+05
1.50	0.66667	200.0	4.15401E+05
1.50	0.57143	50.0	4.22223E+05
1.50	0.57143	100.0	4.18046E+05
1.50	0.57143	150.0	4.16599E+05
1.50	0.57143	200.0	4.15866E+05
2.00	0.66667	50.0	4.37681E+05
2.00	0.66667	100.0	4.34175E+05
2.00	0.66667	150.0	4.32972E+05
2.00	0.66667	200.0	4.32363E+05
2.00	0.57143	50.0	4.39464E+05
2.00	0.57143	100.0	4.35116E+05
2.00	0.57143	150.0	4.33610E+05
2.00	0.57143	200.0	4.32847E+05
2.50	0.66667	50.0	4.49051E+05
2.50	0.66667	100.0	4.45454E+05
2.50	0.66667	150.0	4.44220E+05
2.50	0.66667	200.0	4.43595E+05
2.50	0.57143	50.0	4.50880E+05
2.50	0.57143	100.0	4.46419E+05
2.50	0.57143	150.0	4.44875E+05
2.50	0.57143	200.0	4.44091E+05

VF = 7.00000E-01
E1 = 6.00000E+07
RHO1 = 2.42800E-04
NU1 = 2.00000E-01

RO1/RO2	NU1/NU2	E1/E2	C
1.50	0.66667	50.0	4.44933E+05
1.50	0.66667	100.0	4.41790E+05
1.50	0.66667	150.0	4.40691E+05
1.50	0.66667	200.0	4.40131E+05
1.50	0.57143	50.0	4.46555E+05
1.50	0.57143	100.0	4.42667E+05
1.50	0.57143	150.0	4.41292E+05
1.50	0.57143	200.0	4.40588E+05
2.00	0.66667	50.0	4.57832E+05
2.00	0.66667	100.0	4.54598E+05
2.00	0.66667	150.0	4.53467E+05
2.00	0.66667	200.0	4.52891E+05
2.00	0.57143	50.0	4.59501E+05
2.00	0.57143	100.0	4.55500E+05
2.00	0.57143	150.0	4.54085E+05
2.00	0.57143	200.0	4.53361E+05
2.50	0.66667	50.0	4.66132E+05
2.50	0.66667	100.0	4.62839E+05
2.50	0.66667	150.0	4.61688E+05
2.50	0.66667	200.0	4.61102E+05
2.50	0.57143	50.0	4.67831E+05
2.50	0.57143	100.0	4.63758E+05
2.50	0.57143	150.0	4.62317E+05
2.50	0.57143	200.0	4.61580E+05

VF = 8.00000E-01
E1 = 6.00000E+07
RHO1 = 2.42800E-04
NU1 = 2.00000E-01

RO1/RO2	NU1/NU2	E1/E2	C
1.50	0.66667	50.0	4.67157E+05
1.50	0.66667	100.0	4.63924E+05
1.50	0.66667	150.0	4.62751E+05
1.50	0.66667	200.0	4.62143E+05
1.50	0.57143	50.0	4.68762E+05
1.50	0.57143	100.0	4.64837E+05
1.50	0.57143	150.0	4.63388E+05
1.50	0.57143	200.0	4.62633E+05
2.00	0.66667	50.0	4.75729E+05
2.00	0.66667	100.0	4.72438E+05
2.00	0.66667	150.0	4.71242E+05
2.00	0.66667	200.0	4.70624E+05
2.00	0.57143	50.0	4.77363E+05
2.00	0.57143	100.0	4.73367E+05
2.00	0.57143	150.0	4.71891E+05
2.00	0.57143	200.0	4.71122E+05
2.50	0.66667	50.0	4.81105E+05
2.50	0.66667	100.0	4.77776E+05
2.50	0.66667	150.0	4.76567E+05
2.50	0.66667	200.0	4.75942E+05
2.50	0.57143	50.0	4.82758E+05
2.50	0.57143	100.0	4.78716E+05
2.50	0.57143	150.0	4.77223E+05
2.50	0.57143	200.0	4.76446E+05

VF = 5.00000E-01
E

TIME: 5.59 SECS.

APPENDIX XII

DESCRIPTION AND COMPUTER PROGRAM FOR THE DETERMINATION OF Ω_1, Ω_2, AND Ω_3 IN A HEXAGONAL, MULTIFIBER ELEMENT

DESCRIPTION OF THE COMPUTER PROGRAM

The program is written in FORTRAN 63 (CDC version of FORTRAN IV) for the Control Data Computer Mod 3600. The program is a straight forward calculation of the coefficients in the Airy Function, an evaluation of displacements and stresses along the interface and hexagon boundary and then some double integrations to obtain Ω_1, Ω_2 and Ω_3.

The program consists of the following main parts.

1. A main program, calling the three major subroutines, which are:
2. PTMATCH
3. CHECK
4. OMEGAS

Below is a description of these individual subroutines and their function.

<u>Subroutine PTMATCH</u>

PTMATCH: Reads the input parameters.

Calculates the matrix elements for the 54 equations with the 27 unknowns.

Forms the normal equations.

Solves the normal equations in double precision.

Back substitutes these solutions into the
54 original equations.

PTMATCH uses the following subroutines:

LISTARAY a subroutine to print and label matrices

SOLVE a subroutine which solves linear systems of equations by iterating on the residuals.

Since SOLVE is a useful general purpose subroutine, its usage is described here.

CALL SOLVE (A,B,X,MM,ITER)

where the arguments have the following meaning:

A: Square matrix which contains in single precision the coefficients.

B: A vector with the right hand side of the equations.

X: Contains, after return, the solution in single precision.

MM: The order of the matrix A.

ITER: Number of iterations.

Presently, SOLVE assumes the matrix A to be of dimension (27,27). It also performs exactly ITER iterations.

SOLVE uses DPMATS for the solution of the linear equations in double precision.

It then calculates the residuals in double precision and solves the system again, using the residuals as right hand sides.

This procedure works in the following way:

Let

$$\sum_{k=1}^{n} a_{ik}x_k - bi = 0 \qquad i = 1 \ldots n$$

be the original system to be solved.

Let $\tilde{X}$ be an approximate solution vector, then

$$\sum_{k=1}^{n} a_{ik}\tilde{X}_k - b_i = r_i \qquad i = 1 \ldots n$$

r = residual vector

Now, try to improve the vector $\tilde{x}$ by a Δx so that

$$\sum_{k=1}^{n} a_{ik} (\tilde{x}_k + \Delta x_k) - bi = 0$$

Perform the following operation

$$\Sigma a_{ik} \tilde{x}_k - b_i = r_i$$

$$- \quad \Sigma a_{ik} \tilde{x}_k + \Sigma a_{ik} \Delta x_k = bi$$

$$= \quad \Sigma a_{ik} \Delta x_k = -r_i$$

which means that the correction Δx_k can be obtained by solving this latter system.

SUBROUTINE CHECK

This subroutine calculates displacements and stresses along the interface and the hexagon boundary, not only at the points used in PTMATCH, but also at points between. Check used the subroutines DISPL with the entry points UR1, UTH1, UTH2, UR2 and STRESS with the entry points SR2, STH2, TAU2 SR1, STH1 and TAU1. The two routines DISPL and STRESS are also used by the subroutine OMEGAS. A list of the different entry points and their function is given:

DISPL (U,R,THETA)	input:	R, Theta
	output:	U

ENTRY POINT	U
UR1	U_r^I
UTH1	U_θ^I
UTH2	U_θ^{II}
UR2	U_r^{II}
UR1DR	$\partial U_r^I/\partial r$
UTH1DR	$\partial U_\theta^I/\partial r$

UR1DTH	$\partial U_r^{I}/\partial\theta$
UR2DR	$\partial U_r^{II}/\partial r$
UTH2DR	$\partial U_\theta^{II}/\partial r$
UR2DTH	$\partial U_r^{II}/\partial\theta$
UTH2DTH	$\partial U_\theta^{II}/\partial\theta$

STRESS (R, THETA, SIGMA) input: R, THETA

output: SIGMA

ENTRY POINT	SIGMA
SR2	σ_r^{II}
STH2	σ_θ^{II}
TAU2	$\sigma_{r\theta}^{II}$
SR1	σ_r^{I}
STH1	σ_θ^{I}
TAU1	$\sigma_{r\theta}^{I}$

Subroutine OMEGAS

Calculates Ω_1, Ω_2 and Ω_3 using equations 81, 82 and 83 as given in the Appendix A. It uses the double integration routine DOUBLE, whose usage is described in the listing of the source program. The external subroutines F1, F2, W1BAR and W2BAR represent the integrands.

4. USAGE OF THE PROGRAM

The program requires the following input:

CARD 1 : v_F, a, E^I, E^{II}, ν^I, ν^{II}, ϵ_z, ρ^I

(Format 8E10.5)

where

v_F = percentage of fiber (e.g. 0.65)

a = radius of fiber in inches (e.g. 0.0025)

E^I = Young's modulus for fiber in lbs/sq. inch (e.g. 10 000 000)

E^{II} = Young's modulus for resin in lbs/sq. inch (e.g. 380 000)

ν^I = Poisson constant fiber (e.g. 0.2)

ν^{II} = Poisson constant resin (e.g. 0.35)

ϵ_z = strain in z direction (e.g. -1.)

ρ^I = density of fiber (slugs/inch^3) (e.g. 2.42754 10^{-4})

CARD 2 : ρ^{II}

(Format E10.5)

where ρ^{II} = density of resin (slugs/inch^3) (e.g. 1.15942 10^{-4})

CARD 3 : ITER

(Format I10)

where ITER is the number of iterations in the solution of the normal equation (2 or 3 is enough).

```
      PROGRAM HEXAGON
C
C     THIS PROGRAM COMBINES THE EARLIER PROGRAMS  ** PTMATCH **
C     ** PTMATCH2 **   AND ** OMEGAS ** INTO ONE SINGLE PROGRAM
C
C
C     PTMATCH READS THE PARAMETERS AND THEN FINDS THE COEFFICIENTS
C     OF THE AIRY FUNCTION
C
      CALL PTMATCH
C
C
C     CHECK CALCULATES DISPLACEMENTS AND STRESSES ALONG THE INTERFACE
C     AND THE HEXAGON BOUNDARY
C
      CALL CHECK
C
C
      CALL OMEGAS
C
C     CALCULATES THE THREE VALUES  OMEGA 1, OMEGA 2, AND OMEGA 3
      STOP
      END
```

```
      SUBROUTINE PTMATCH
C
C     PROGRAM FINDS AND PUNCHES THE COEFFICIENTS FOR THE PLAIN STRAIN
C     PROBLEM WITH A HEXAGONAL FIBER ARRANGEMENT.
C
C     IT USES 9 POINTS AT THE INTERFACE AT THE ANGLES OF
C                   THETA = 0
C                   THETA = PI/48
C                   THETA = 2*PI/48
C                   THETA = ......
C                   THETA = PI/6
C
C     AND 9 POINTS AT THE HEXAGON BOUNDARY AT THE SAME ANGLES.
C
C
C     THE EQUATIONS EXPRESSING DISPLACEMENTS ARE ALL MULTIPLIED (WEIGHTED)
C     WITH E1
C
C     DOUBLE PRECISION IS USED TO SOLVE THE NORMAL EQUATIONS
C
C
      COMMON / INP1/ VF,AA,E1,E2,FNU1,FNU2,EZ,BB,PI
      COMMON/COEFCT/ AN2(4),BN2(4),CN2(4),DN2(4),AN1(4),CN1(4),
     *               BO2,CO2,CO1
      COMMON/RHOS/ RHO1,RHO2
      DIMENSION A(54,27),ATR(27,54),AMT(27,27),RHS(27),B(54),SCR(27)
C
C *** INPUT DATA
      PI = 4.*ATAN(1.)
      READ 883,VF,AA,E1,E2,FNU1,FNU2,EZ,RHO1,RHO2
  883 FORMAT(8E10.5)
      READ 884,ITER
  884 FORMAT(8I10)
      BB = AA*SQRT(PI*SQRT(3.)/(6.*VF))
      PRINT 995,VF,AA,E1,E2,FNU1,FNU2,EZ,RHO1,RHO2,BB
  995 FORMAT (11H INPUT DATA     ///
     1 7H VF     =   E15.5 /
     2 7H A      =   E15.5 /
     3 7H E1     =   E15.5 /
     4 7H E2     =   E15.5 /
     5 7H NU 1  =   E15.5 /
     6 7H NU 2  =   E15.5 /
     6 7H EZ     =   E15.5 /
     7 7H RHO 1= E15.5/
     8 7H RHO 2= E15.5/
     7 7H B      =   E15.5 /                   ///)
     °                                     MM = 27
C
C
      DO 50 J = 1,27
      DO 50 I = 1,54
   50 A(I,J) = 0.0
      DO 55 I = 1,54
   55 B(I) = 0.0
C
C *** EQUATIONS 1 THRU 9                        SIGMA R 2 - SIGMA R 1 = 0
C
C
      R = AA
      DO 200 I = 1,9
      FI = I
      THETA = (FI-1.)*PI/48.
```

```
C
      DO 100 NN = 1,4
      FN = 6*NN   $   N = 6*NN
      ICOL = (NN-1)*4
      CSN = COS(FN*THETA)
      IF (ABS(CSN).LT.1.E-7) CSN = 0.0
C
      A(I,ICOL+1) = -FN*(FN-1.)*R**(N-2)*CSN
      A(I,ICOL+2) = -FN*(FN+1.)*R**(-N-2)*CSN
      A(I,ICOL+3) = -(FN+1.)*(FN-2.)*R**N*CSN
      A(I,ICOL+4) = -(FN-1.)*(FN+2.)*R**(-N)*CSN
C
      ICOL = (NN-1)*2
      A(I,ICOL+17)=FN*(FN-1.)*R**(N-2)*CSN
      A(I,ICOL+18)=(FN+1.)*(FN-2.)*R**N*CSN
C
  100 CONTINUE
      A(I,25) = R**(-2)
      A(I,26) = 2.
      A(I,27) = -2.
  200 CONTINUE
C
C *** EQUATIONS 10 THRU 18                              UR 2 - UR 1 = 0
C
      EE1 = (1.+FNU1)/E1   $   EE2 = (1.+FNU2)/E2
C
      DO 400 I = 1,9
      II = I+9
      R= AA
      FI = I
      THETA =(FI-1.)*PI/48.
C
      DO 300 NN = 1,4
      FN = 6*NN
      N=6*NN
      ICOL = (NN-1)*4
      CSN = COS(FN*THETA)
      IF (ABS(CSN).LT.1.E-7) CSN = 0.0
C
      A(II,ICOL+1)=-FN*R**(N-1)*CSN*EE2
      A(II,ICOL+2) = FN*R**(-N-1)*CSN*EE2
      A(II,ICOL+3) = -(FN-2.+4.*FNU2)*R**(N+1)*CSN*EE2
      A(II,ICOL+4)=(FN+2.-4.*FNU2)*R**(-N+1)*CSN *EE2
C
      ICOL = (NN-1)*2
      A(II,ICOL+17) = FN*R**(N-1)*EE1*CSN
      A(II,ICOL+18)=(FN-2.+4.*FNU1)*R**(N+1)*CSN*EE1
C
  300 CONTINUE
      A(II,25) = -R**(-1)*EE2
      A(II,26)=2.*(1.-2.*FNU2)*R*EE2
      A(II,27)=-2.*(1.-2.*FNU1)*R*EE1
C
      B(II)=-FNU2*R*EZ + FNU1*R*EZ
  400         CONTINUE
C
C
C     MULTIPLY EQUATIONS 10 THRU 18 BY E1
C
      DO 425 I = 10,18
      DO 420 J = 1,27
  420 A(I,J) = A(I,J)*E1
```

```
  425 B(I) = B(I)*E1
C
C
C *** EQUATIONS 19 THRU 27                                     TAU 2 - TAU 1 = 0
C
C
      DO 600 I = 1,9
      II = I + 18
      R= AA
      FI = I
      THETA = (FI-1.)*PI/48.
C
      DO 500 NN = 1,4
      FN = 6.*NN   $   N = 6*NN
      ICOL = (NN-1)*4
      SSN = SIN(FN*THETA)
      IF (ABS(SSN) .LT. 1.E-7)   SSN = 0.0
C
      A(II,ICOL+1) = FN*(FN-1.)*R**(N-2)*SSN
      A(II,ICOL+2) = -FN*(FN+1.)*R**(-N-2)*SSN
      A(II,ICOL+3) = FN*(FN+1.)*R**N*SSN
      A(II,ICOL+4) = -FN*(FN-1.)*R**(-N)*SSN
C
      ICOL = (NN-1)*2
      A(II,ICOL+17) = -FN*(FN-1.)*R**(N-2)*SSN
      A(II,ICOL+18) = -FN*(FN+1)*R**N*SSN
C
  500 CONTINUE
C
  600 CONTINUE
C
C
C *** EQUATIONS 28 THRU 36                           U THETA 2 - UTHETA 1 = 0
C
C
      DO 800 I = 1,9
      II = I + 27
      R = AA
       FI = I
      THETA =(FI-1)*PI/48.
C
      DO 700 NN = 1,4
      FN = 6.*NN   $ N = 6*NN
      ICOL =(NN-1)*4
      SSN = SIN(FN*THETA)
      IF (ABS(SSN) .LT. 1.E-7)   SSN = 0.0
C
      A(II,ICOL + 1)=FN*R**(N-1)*SSN*EE2
      A(II,ICOL+2) = FN*R**(-N-1)*SSN*EE2
      A(II,ICOL + 3) = (FN+4.-4.*FNU2)*R**(N+1)*SSN*EE2
      A(II,ICOL+4) = (FN-4.+4.*FNU2)*R**(-N+1)*SSN*EE2
C
      ICOL = (NN-1)*2
      A(II,ICOL+17)=-FN*R**(N-1)*SSN*EE1
      A(II,ICOL+18)=-(FN+4.-4.*FNU1)*R**(N+1)*SSN*EE1
C
  700 CONTINUE
C
  800 CONTINUE
C
C
C     MULTIPLY EQUATIONS 28 THRU 36 BY E1
```

```
C
      DO 825 I = 28,36
      DO 820 J = 1,27
  820 A(I,J) = A(I,J)*E1
  825 B(I) = B(I) * E1
C
C
C     EQUATIONS 37 THRU 45  U N  AT HEXAGON BOUNDARY  EQUALS ZERO
C
C     FIRST PART       U R * COS(PI/6-THETA)
C
C
      DO 1000 I = 1,9
      FI = I-1
      II = I+36
      R = BB/COS(PI/6.-FI*PI/48.)
      THETA = FI*PI/48.
C
      CTS = COS(PI/6.-THETA)   $   STS=SIN(PI/6.-THETA)
      IF (ABS(CTS).LT.1.E-7)   CTS = 0.0
      IF (ABS(STS).LT.1.E-7)   STS = 0.0
C
      DO 900 NN = 1,4
      FN = 6*NN  $  N = 6*NN
      ICOL =(NN-1)*4
      SSN = SIN (FN*THETA)
      IF (ABS(SSN) .LT. 1.E-7)   SSN = 0.0
      CSN = COS(FN*THETA)
      IF (ABS(CSN).LT.1.E-7) CSN = 0.0
C
      A(II,ICOL+1)=-FN*R**(N-1)*EE2*CSN*CTS
      A(II,ICOL+2)=FN*R**(-N-1)*EE2*CSN*CTS
      A(II,ICOL+3)=-(FN-2.+4.*FNU2)*R**(N+1)*EE2*CSN*CTS
      A(II,ICOL+4)=(FN+2.-4.*FNU2)*R**(-N+1)*EE2*CSN*CTS
C
  900 CONTINUE
C
      A(II,25) = -R**(-1)*EE2*CTS
      A(II,26)=2.*(1.-2.*FNU2)*R*EE2*CTS
C
      B(II) = -FNU2*FZ*R*CTS
C
 1000 CONTINUE
C
C     SECOND PART    U THETA * SIN (PI/6 - THETA)
C
      DO 1200 I = 1,9
      FI = I-1
      II = I + 36
      R = BB/COS(PI/6.-FI*PI/48.)
      THETA = FI*PI/48.
      CTS = COS(PI/6.-THETA)  $ STS = SIN (PI/6. - THETA)
      IF (ABS(CTS).LT.1.E-7)   CTS = 0.0
      IF (ABS(STS).LT.1.E-7)   STS = 0.0
C
      DO 1100 NN = 1,4
      FN = 6*NN  $  N = 6*NN
C
      ICOL =(NN-1)*4
      SSN = SIN(FN*THETA)
      IF (ABS(SSN) .LT. 1.E-7)   SSN = 0.0
      CSN = COS(FN*THETA)
```

```
      IF (ABS(CSN).LT.1.E-7) CSN = 0.0
C
      A(II,ICOL+1)=A(II,ICOL+1)+FN*R**(N-1)*EE2*SSN*STS
      A(II,ICOL+2) = A(II,ICOL+2) + FN*R**(-N-1)*EE2*SSN*STS
      A(II,ICOL+3)=A(II,ICOL+3)+(FN+4.-4.*FNU2)*R**(N+1)*EE2*SSN*STS
      A(II,ICOL+4)=A(II,ICOL+4)+(FN-4.+4.*FNU2)*R**(-N+1)*EE2*SSN*STS
C
 1100 CONTINUE
C
 1200 CONTINUF
C
C
C     MULTIPLY EQUATIONS 37 THRU 45 BY E1
      DO 1225 I = 37,45
      DO 1220 J = 1,27
 1220 A(I,J) = A(I,J) * E1
 1225 B(I) = B(I) * E1
C
C     EQUATIONS 46 - 54 TAU NT AT HEXAGON BOUNDARY EQUALS ZERO
C
C     FIRST PART -0.5*SIGMAR*SIN(PI/3-2*THETA)
C
      DO 1400 I = 1,9
      FI = I - 1
      II = I+45
      R = BB/COS(PI/6.-FI*PI/48.)
      THETA = FI*PI/48.
      STN = -0.5*SINF(PI/3.-2.*THETA)
      IF (ABS(STN).LT.1.E-7)  STN = 0.0
C
      DO 1300 NN = 1,4
      FN = 6.*NN  $  N = 6*NN
      ICOL = (NN-1) *4
      SSN = SIN(FN*THETA)
      IF (ABS(SSN) .LT. 1.E-7)   SSN = 0.0
      CSN = COS(FN*THETA)
      IF (ABS(CSN).LT.1.E-7) CSN = 0.0
C
C
      A(II,ICOL+1) =-FN*(FN-1.)*R**(N-2)*CSN*STN
      A(II,ICOL+2)=-FN*(FN+1.)*R**(-N-2)*CSN*STN
      A(II,ICOL+3)=-(FN+1.)*(FN-2.)*R**N*CSN*STN
      A(II,ICOL+4)=-(FN-1)*(FN+2.)*R**(-N)*CSN*STN
C
 1300 CONTINUE
C
      A(II,25) = R**(-2)*STN
      A(II,26) = 2.*STN
 1400 CONTINUE
C
C
C     SECOND PART.. SIGMA THETA *0.5 * SIN(PI/3. -2*THETA)
C
      DO 1600 I = 1,9
      FI = I-1
      II = I + 45
      R = BB/COS(PI/6.-FI*PI/48.)
      THETA = FI*PI/48.
      STN = 0.5*SIN(PI/3.-2.*THETA)
      IF (ABS(STN).LT.1.E-7)  STN = 0.0
C
      DO 1500 NN = 1,4
```

```
      FN = 6*NN  $ N = 6*NN
      ICOL = (NN-1)*4
      SSN = SIN(FN*THETA)
      IF (ABS(SSN) .LT. 1.E-7)   SSN = 0.0
      CSN = COS(FN*THETA)
      IF (ABS(CSN).LT.1.E-7) CSN = 0.0
C
      A(II,ICOL+1) = A(II,ICOL+1)+FN*(FN-1.)*R**(N-2)*CSN*STN
      A(II,ICOL+2)=A(II,ICOL+2)+FN*(FN+1.)*R**(-N-2)*CSN*STN
      A(II,ICOL+3) = A(II,ICOL+3)+(FN+1.)*(FN+2.)*R**N*CSN*STN
      A(II,ICOL+4)=A(II,ICOL+4)+(FN-1.)*(FN-2.)*R**(-N)*CSN*STN
C
 1500 CONTINUE
C
      A(II,25) = A(II,25) - R**(-2)       * STN
      A(II,26) = A(II,26) + 2.*STN
C
 1600 CONTINUE
C
C
C     THIRD PART   TAU R THETA *COS(PI/3 -2*THETA)
C
      DO 1800 I = 1,9
      II = I+45
      FI = I-1
      R = BB/COS(PI/6.-FI*PI/48.)
      THETA = FI*PI/48.
      CTN = COS(PI/3.-2.*THETA)
      IF (ABS(CTN).LT. 1.E-7)  CTN = 0.0
C
      DO 1700 NN = 1,4
      FN = 6*NN  $ N = 6*NN
      ICOL = (NN-1) *4
      SSN = SIN(FN*THETA)
      IF (ABS (SSN).LT.1.E-7)  SSN = 0.0
      CSN = COS(FN*THETA)
      IF (ABS(CSN).LT.1.E-7) CSN = 0.0
C
      A(II,ICOL+1)=A(II,ICOL+1)+FN*(FN-1)*R**(N-2)*SSN*CTN
      A(II,ICOL+2)=A(II,ICOL+2)-FN*(FN+1.)*R**(-N-2)*SSN*CTN
      A(II,ICOL+3)=A(II,ICOL+3)+FN*(FN+1)*R**N*SSN*CTN
      A(II,ICOL+4)=A(II,ICOL+4)-FN*(FN-1.)*R**(-N)*SSN*CTN
C
 1700 CONTINUE
C
 1800 CONTINUE
C
C
C     FORM A TRANSPOSE
C
      DO 2000 I = 1,27
      DO 2000 J = 1,54
 2000 ATR(I,J) = A(J,I)
C
C     FORM NORMAL EQUATIONS BY PREMULTIPLYING WITH A TRANSPOSE
C
      DO 2120 I = 1,27
      DO 2120 J = 1,27
      DO 2120 K=1,54
C
 2120 AMT(I,J)   = AMT(I,J)+ATR(I,K)*A(K,J)
      DO 2130  I = 1,27
```

```
      DO 2130 K = 1,54
 2130 RHS(I) = RHS(I)+ATR(I,K)*B(K)
C
C     PRINT MATRIX
      CALL LISTARAY(A,54,54,27,1)
C
C     PRINT MATRIX OF NORMAL EQUATIONS
      CALL LISTARAY (AMT,27,27,27,1)
      CALL LISTARAY(RHS,27,27,1,1)
C
C
C     SOLVE NORMAL EQUATIONS USING DOUBLE PRECISION AND ITERATING ON THE
C     RESIDUALS
C
      CALL SOLVE (AMT,RHS,SCR,MM,ITER)
      DO 3335 I = 1,MM
 3335 RHS(I) = SCR(I)
      DO 2450 N = 1,4
      NN = (N-1)*4    $    NNN = (N-1)*2
      AN2(N) = RHS(1+NN)
      BN2(N) = RHS(2+NN)
      CN2(N) = RHS(3+NN)
      DN2(N) = RHS(4+NN)
      AN1(N) = RHS(17+NNN)
      CN1(N) = RHS(18+NNN)
 2450 CONTINUE
      B02=RHS(25)
      C02 = RHS(26)
      C01 = RHS(27)
  992 FORMAT(I10,F20.6)
C
C
C *** BACKSUBSTITUE INTO THE ORIGINAL 54 EQUATIONS
C
      PRINT 999
  999 FORMAT(1H1)
      PRINT 997
  997 FORMAT(' BACK SUBSTITUTION INTO 54 ORIGINAL EQUATIONS'/
     *9X,'I',10X,'RHS',16X,'BACK'//)
      DO 2300 K = 1,54
      BACK = 0.0
      DO 2225 J = 1,27
 2225 BACK = BACK + RHS(J)*A(K,J)
      PRINT 994, K, B(K), BACK
  994 FORMAT(I10,2E20.4)
 2300 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE CHECK
C
C     THIS PROGRAM CHECKS THE BOUNDARY CONDITIONS IN THE POINT MATCHING PROBLEM
C
C
      COMMON / INPT/ VF,AA,E1,E2,FNU1,FNU2,EZ,BB,PI
      COMMON/COEFCT/ AN2(4),BN2(4),CN2(4),DN2(4),AN1(4),CN1(4),
     *               B02,C02,C01
      EQUIVALENCE (A,AA),(B,BB)
C
C     CHECK THE DISPLACEMENTS
C
C
      PRINT 995
  995 FORMAT(1H1)
C
C *** TEST AT INTERFACE
C
      DELTA = PI/96.
      R = A
      PRINT 992
  992 FORMAT(' TEST DISPLACEMENTS AT INTERFACE'//
     *7X,'THETA',8X,'UR 1',12X,'UR 2',6X,'U THETA 1',5X,'U THETA 2'///)
      DO 100 I = 1,17
      FI = I-1
      THETA = FI *DELTA
      CALL UR1(U1,R,THETA)
      CALL UR2(U2,R,THETA)
      CALL UTH1(U3,R,THETA)
      CALL UTH2(U4,R,THETA)
      PRINT 991,THETA,U1,U2,U3,U4
  991 FORMAT (5F15.6)
  100 CONTINUE
C
C     CHECK AT HEXAGON BOUNDARY
C
      PRINT 995
      PRINT 996
  996 FORMAT(' TEST DISPLACEMENTS AT HEXAGON BOUNDARY'//
     *7X,'THETA',10X,'R',13X,'UR 2',10X,'U THETA 2',6X,'U NORMAL 2',//)
      DO 200 I = 1,17
      FI = I-1
      THETA = FI * DELTA
      R = B/COS(PI/6.-THETA)
      CALL UR2(U2,R,THETA)
      CALL UTH2(U4,R,THETA)
      UN = U2*COS(PI/6.-THETA) + U4*SIN(PI/6.-THETA)
      PRINT 991,THETA,R,U2,U4,UN
  200 CONTINUE
C
C
C      CHECK THE STRESSES
C
C  *  TEST AT INTERFACE
C
      PRINT 995
      PRINT 890
  890 FORMAT(' TEST STRESSES AT INTERFACE'//)
      R = A
      DO 300 I = 1,17
      FI = I - 1
      THETA = FI * DELTA
```

```
      CALL SR1(R,THETA, S1)
      CALL STH1(R,THETA,S1)
      CALL STH1(R,THETA,S2)
      CALL TAU1(R,THETA,S3)
      CALL SR2(R,THETA,S4)
      CALL STH2(R,THETA,S5)
      CALL TAU2(R,THETA,S6)
      PRINT 997, THETA,S1,S2,S3,S4,S5,S6
  997 FORMAT(' THETA=',E15.4,3X,'S R 1 =',E15.4,3X,'S THETA 1 =',E15.4,
     *3X,'TAU 1 =',E15.4/
     *25X,'S R 2 =',E15.4,3X,'S THETA 2 =',E15.4,
     *3X,'TAU 2 =',E15.4//)
C
  300 CONTINUE
C
C
C     TEST AT HEXAGON BOUNDARY
C
      PRINT 995
      PRINT 891
  891 FORMAT(' TEST STRESSES AT HEXAGON BOUNDARY'//)
      DO 400 I = 1,17
      FI = I-1
      THETA = FI * DELTA
      R = B/COS(PI/6. - THETA)
      CALL SR2(R,THETA,S1)
      CALL STH2(R,THETA, S2)
      CALL TAU2(R,THETA,S3)
      TAUNORM=-0.5*(S1-S2)*SIN(PI/3.-2.*THETA)+S3*COS(PI/3.-2.*THETA)
      PRINT 993,THETA,R,S1,S2,S3,TAUNORM
  993 FORMAT(' THETA =',E15.4,3X,'R =',E15.4,3X,'SIGMA P 2 =',E15.4,
     *3X,'SIGMA THETA 2 =',E15.4/
     *26X,' TAU 2 =',E15.4,3X,'TAU NORMAL =',E15.4//)
  400 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE OMEGAS
C
C     CALCULATES OMEGA 1 OMEGA2  AND OMEGA 3 BY DOUBLE INTEGRATION
C     USING EQUATIONS 81,  82,   AND 83.
C
      EXTERNAL F1,F2
      EXTERNAL W1BAR,W2BAR
C
      EQUIVALENCE (B,BB)
      EQUIVALENCE (A,AA)
      COMMON / INPT/ VF,AA,E1,E2,FNU1,FNU2,EZ,BB,PI
      COMMON/RHOS/ RHO1,RHO2
C
C
      CALL DOUBLE (0.,PI/6.,0.,A,1.E-6,20,RES1,INTX1,R1,F1)
      PART1 = RES1*RHO1
      CALL DOUBLE (0.0,PI/6.,A,B/COS(PI/12.),1.E-6,20,RES2,INTX2,R2,F2)
      PART2 = RES2*RHO2
      PRINT 995
  995 FORMAT(1H1)
      OM1 = PART1 + PART2
      OM1 = OM1 * 12.
      PRINT 991,OM1
  991 FORMAT(///,8H OMEGA1=     E20.5)
      OMEGA2 = RHO1*A**2*PI/24. + RHO2*(B**2/SQRT(3.)-A**2*PI/6.)  /4.
      OMEGA2 = OMEGA2 * 12.
      PRINT 994,OMEGA2
  994 FORMAT(///,8H OMEGA2=     F20.5)
C
C     CALCULATE OMEGA3
C
      TEIL1 = F1*A**2*PI/24.
      TEIL2=F2/4.*(B**2/SQRT(3.) -A**2*PI/6.)
      CALL DOUBLE(0.,PI/6.,0.,A,1.E-4,20,RES3,INTX3,R3,W1BAR)
      CALL DOUBLE(0.,PI/6.,A,B/COS(PI/12.),1.E-4,20,RES4,INTX4,R4,W2BAR)
      OMEGA3=TEIL1+TEIL2+RES3+RES4
      OMEGA3 = OMEGA3 * 12.
      PRINT 998,OMEGA3
  998 FORMAT(///8H OMEGA3=,  F20.5)
      RETURN
      END
```

```
      SUBROUTINE DISPL(U,R,THETA)
      COMMON / INPT/ VF,AA,E1,E2,FNU1,FNU2,EZ,BB,PI
      COMMON/COEFCT/ AN2(4),BN2(4),CN2(4),DN2(4),AN1(4),CN1(4),
     *               B02,C02,C01
      ENTRY UR1
      U = 0.0
      DO 100 NN = 1,4
      N = 6*NN
      FN = N
      U=U-(AN1(NN)*FN*R**(N-1) + CN1(NN)*(FN-2.+4.*FNU1)*R**(N+1))
     1              * COS(FN*THETA)
  100 CONTINUE
      U = U + C01*2.*(1.-2.*FNU1)*R
      U = U*(1.+FNU1)/E1
      U = U + FNU1*EZ*R
      RETURN
C
C
      ENTRY UTH1
      U = 0.0
      DO 200 NN = 1,4
      N = 6*NN
      FN = N
      T1 = FN*AN1(NN)*R**(N-1)
      T2 = (FN +4.-4.*FNU1)*CN1(NN)*R**(N+1)
      U = U +(T1+T2)*SIN(FN*THETA)
  200 CONTINUE
      U = U*(1.+FNU1) / E1
      RETURN
C
C
      ENTRY UTH2
      U = 0.0
      DO 300 NN = 1,4
      N = 6*NN
      FN = N
      T1 =  FN*AN2(NN)*R**(N-1)
      T2 = FN*BN2(NN)*R**(-N-1)
      T3 = (FN+4.-4.*FNU2)*CN2(NN)*R**(N+1)
      T4 = (FN-4.+4.*FNU2)*DN2(NN)*R**(-N+1)
      U = U+(T1+T2+T3+T4)*SIN(FN*THETA)
  300 CONTINUE
      U = U*(1.+FNU2)/E2
      RETURN
C
C
      ENTRY UR2
      U = 0.0
      DO 400 NN = 1,4
      N = 6*NN
      FN = N
      T1 = FN*AN2(NN)*R**(N-1)
      T2 = -FN*BN2(NN)*R**(-N-1)
      T3 =(FN-2.+4.*FNU2)*CN2(NN)*R**(N+1)
      T4 = -(FN+2.-4.*FNU2)*DN2(NN)*R**(-N+1)
      U = U -(T1+T2+T3+T4)    * COS(FN*THETA)
  400 CONTINUE
      U = U + 2.*(1.-2.*FNU2)*C02*R
      U = U -B02*R**(-1)
      U = U*(1.+FNU2)/E2
      U = U + FNU2*EZ*R
      RETURN
```

```
      ENTRY UR1DR
      U=0.0
      DO 500 NN=1,4
      N=6*NN $ FN=N
      T1=FN*(FN-1.)*AN1(NN)*R**(N-2)
      T2=(FN-2.+4.*FNU1)*(FN+1.)*CN1(NN)*R**N
      U=U-(T1+T2)*COS(FN*THETA)
  500 CONTINUE
      U=U+2.*(1.-2.*FNU1)*C01
      U=U*(1.+FNU1)/E1
      U=U+FNU1*EZ
      RETURN
C
C
      ENTRY UTH1DR
      U=0.0
      DO 600 NN=1,4
      N=6*NN $ FN=N
      T1=FN*(FN-1.)*AN1(NN)*R**(N-2)
      T2=(FN+4.-4.*FNU1)*(FN+1.)*CN1(NN)*R**N
      U=U+(T1+T2)*SIN(FN*THETA)
  600 CONTINUE
      U=U*(1.+FNU1)/E1
      RETURN
C
C
      ENTRY UR1DTH
      U=0.0
      DO 700 NN=1,4
      N=6*NN $ FN=N
      T1=FN*AN1(NN)*R**(N-1)
      T2=(FN-2.+4.*FNU1)*CN1(NN)*R**(N+1)
      U=U+(T1+T2)*FN*SIN(FN*THETA)
  700 CONTINUE
      U=U*(1.+FNU1)/E1
      RETURN
C
C
      ENTRY UTH1DTH
      U=0.0
      DO 800 NN=1,4
      N=6*NN $ FN=N
      T1=FN*AN1(NN)*R**(N-1)
      T2=(FN+4.-4.*FNU1)*CN1(NN)*R**(N+1)
      U=U+(T1+T2)*FN*COS(FN*THETA)
  800 CONTINUE
      U=U*(1.+FNU1)/E1
      RETURN
C
C
      ENTRY UR2DR
      U=0.0
      DO 900 NN=1,4
      N=6*NN $ FN=N
      T1=FN*(FN-1.)*AN2(NN)*R**(N-2)
      T2=-FN*(-FN-1.)*BN2(NN)*R**(-N-2)
      T3=(FN-2.+4.*FNU2)*(FN+1.)*CN2(NN)*R**N
      T4=-(FN+2.-4.*FNU2)*(-FN+1.)*DN2(NN)*R**(-N)
      U=U-(T1+T2+T3+T4)*COS(FN*THETA)
  900 CONTINUE
      U=U+B02*R**(-2)+C02*2.*(1.-2.*FNU2)
      U=U*(1.+FNU2)/E2 + FNU2*EZ
```

```
      RETURN
C
C
      ENTRY UTH2DR
      U=0.0
      DO 1000 NN=1,4
      N=6*NN $ FN=N
      T1=FN*(FN-1.)*AN2(NN)*R**(N-2)
      T2=FN*(-FN-1.)*BN2(NN)*R**(-N-2)
      T3=(FN+4.-4.*FNU2)*(FN+1.)*CN2(NN)*R**N
      T4=(FN-4.+4.*FNU2)*(-FN+1.)*DN2(NN)*R**(-N)
      U=U+(T1+T2+T3+T4)*SIN(FN*THETA)
 1000 CONTINUE
      U=U*(1.+FNU2)/E2
      RETURN
C
C
      ENTRY UR2DTH
      U=0.0
      DO 1100 NN=1,4
      N=6*NN $ FN=N
      T1=FN*AN2(NN)*R**(N-1)
      T2=-FN*BN2(NN)*R**(-N-1)
      T3=(FN-2.+4.*FNU2)*CN2(NN)*R**(N+1)
      T4=-(FN+2.-4.*FNU2)*DN2(NN)*R**(-N+1)
      U=U+(T1+T2+T3+T4)*FN*SIN(FN*THETA)
 1100 CONTINUE
      U=U*(1.+FNU2)/E2
      RETURN
C
C
      ENTRY UTH2DTH
      U=0.0
      DO 1200 NN=1,4
      N=6*NN $ FN=N
      T1=FN*AN2(NN)*R**(N-1)
      T2=FN*BN2(NN)*R**(-N-1)
      T3=(FN+4.-4.*FNU2)*CN2(NN)*R**(N+1)
      T4=(FN-4.+4.*FNU2)*DN2(NN)*R**(-N+1)
      U=U+(T1+T2+T3+T4)*FN*COS(FN*THETA)
 1200 CONTINUE
      U=U*(1.+FNU2)/E2
      RETURN
      END
```

```
      SURROUTINE STRESS (R,THETA,SIGMA)
C
      COMMON / INPT/ VF,AA,E1,E2,FNU1,FNU2,E2,RB,PI
      COMMON/COEFCT/ AN2(4),BN2(4),CN2(4),DN2(4),AN1(4),CN1(4),
     *               B02,C02,C01
C
      ENTRY SR2
      SIGMA = 0.0
      DO 100 NN = 1,4
      N = 6*NN
      FN = N
      T1 = FN*(FN-1.)*AN2(NN)*R**(N-2)
      T2 = FN*(FN+1.)*BN2(NN)*R**(-N-2)
      T3 = (FN+1.)*(FN-2.)*CN2(NN)*R**N
      T4=(FN-1.)*(FN+2.)*DN2(NN)*R**(-N)
      SIGMA = SIGMA-(T1+T2+T3+T4)*COS(FN*THETA)
  100 CONTINUE
      SIGMA = SIGMA +B02*R**(-2) + C02*2.
      RETURN
C
C
      ENTRY STH2
      SIGMA = 0.0
      DO 200 NN = 1,4
      N = 6*NN
      FN = N
      T1 = FN*(FN-1.)*AN2(NN)*R**(N-2)
      T2=FN*(FN+1.)*BN2(NN)*R**(-N-2)
      T3 = (FN+1.)*(FN+2.)*CN2(NN)*R**N
      T4=(FN-1.)*(FN-2.)*DN2(NN)*R**(-N)
      SIGMA = SIGMA+(T1+T2+T3+T4)*COS(FN*THETA)
  200 CONTINUE
      SIGMA = SIGMA-B02*R**(-2)+2.*C02
      RETURN
C
C
      ENTRY TAU2
      SIGMA = 0.0
      DO 300 NN = 1,4
      N = 6*NN
      FN = N
      T1 = FN*(FN-1.)*AN2(NN)*R**(N-2)
      T2=-FN*(FN+1.)*BN2(NN)*R**(-N-2)
      T3 = FN*(FN+1.)*CN2(NN)*R**N
      T4 = -FN*(FN-1.)*DN2(NN)*R**(-N)
      SIGMA = SIGMA + (T1+T2+T3+T4)*SIN(FN*THETA)
  300 CONTINUE
      RETURN
      ENTRY  SR1
      SIGMA = 0.0
      DO 400 NN = 1,4
      N = 6*NN
      FN = N
      T1 = FN*(FN-1.)*AN1(NN)*R**(N-2)
      T2 = (FN+1.)*(FN-2.)*CN1(NN)*R**(N)
      SIGMA = SIGMA - (T1+T2)*COS(FN*THETA)
  400 CONTINUE
      SIGMA = SIGMA + 2.*C01
      RETURN
C
C
      ENTRY STH1
```

```
      SIGMA = 0.0
      DO 500 NN = 1,4
      N = 6*NN
      FN = N
      T1 = FN*(FN-1)*AN1(NN)*R**(N-2)
      T2=(FN+1.)*(FN+2.)*CN1(NN)*R**N
      SIGMA = SIGMA +(T1+T2)*COS(FN*THETA)
  500 CONTINUE
      SIGMA = SIGMA +2.*C01
      RETURN
C
C
      ENTRY TAU1
      SIGMA = 0.0
      DO 600 NN = 1,4
      N = 6*NN
      FN = N
      T1 = FN*(FN-1.)*AN1(NN)*R**(N-2)
      T2 = FN*(FN+1.)*CN1(NN)*R**N
      SIGMA = SIGMA + (T1+T2)*SIN(FN*THETA)
  600 CONTINUE
      RETURN
      END
```

```
      FUNCTION WBAR(THETA,R)
      ENTRY W1BAR
      IF (R.EQ. 0.0) GO TO 10
      CALL SR1(R,THETA,S1)
      CALL STH1(R,THETA,S2)
      CALL TAU1(R,THETA,S3)
      CALL UR1(U1,R,THETA)
      CALL UTH1(U2,R,THETA)
      CALL UR1DR(DU1,R,THETA)
      CALL UTH1DTH(DU2,R,THETA)
      CALL UTH1DR(DU3,R,THETA)
      CALL UR1DTH(DU4,R,THETA)
      T1 = S1*DU1
      T2= S2*(U1/R+DU2/R)
      T3=S3*(DU3-U2/R+DU4/R)
      WBAR=0.5*R*(T1+T2+T3)
      RETURN
   10 WBAR = 0.
      RETURN
      ENTRY W2BAR
      CALL SR2(R,THETA,S1)
      CALL STH2(R,THETA,S2)
      CALL TAU2(R,THETA,S3)
      CALL UR2(U1,R,THETA)
      CALL UTH2(U2,R,THETA)
      CALL UR2DR(DU1,R,THETA)
      CALL UTH2DTH(DU2,R,THETA)
      CALL UTH2DR(DU3,R,THETA)
      CALL UR2DTH(DU4,R,THETA)
      T1 = S1*DU1
      T2= S2*(U1/R+DU2/R)
      T3=S3*(DU3-U2/R+DU4/R)
      WBAR=0.5*R*(T1+T2+T3)
      RETURN
      END
```

```
      SUBROUTINE SOLVE (A,B,X,MM,ITER)
C
C     THIS SUBROUTINE SOLVES THE LINEAR EQUATIONS  A* X  =  B
C     IT IMPROVES THE SOLUTION BY ITERATING ON THE RESIDUALS
C
C        INPUT      A          ORIGINAL COEFFICIENT MATRIX (SINGLE PREC.)
C                   B          RIGHT HAND SIDE VECTOR (SINGLE PRECISION)
C                   MM         ORDER OF MATRIX
C                   ITER       MAXIMUM NUMBER OF ITERATIONS
C
C        OUTPUT     X          SOLUTION VECTOR (SINGLE PRECISION)
C
      DIMENSION ADP(27,27),ADPS(27,27),BDP(27),BACKDP(27),BDPS(27),
     1          XDP(27),ERRDP(27)
      DIMENSION X(1)
      DIMENSION A(27,27),B(27)
C
      TYPE DOUBLE DETDP,ADP,ADPS,BDP,BACKDP,BDPS,XDP,ERRDP
C
C     SAVE MATRICES AS DOUBLE PRECISION MATRICES
C
      DO 20 I = 1,MM
      DO 10 J = 1,MM
   10 ADPS(I,J) = A(I,J)
      BDPS(I) = B(I)
   20 BDP(I) = B(I)
      DO 30 I = 1,MM
   30 XDP(I) = 0.0
C
C     PERFORM THE ITERATIONS
C
      DO 9999 ICOUNT = 1,ITER
C
      DO 50 I = 1,MM
      DO 40 J = 1,MM
   40 ADP(I,J) = ADPS(I,J)
   50 BACKDP(I) = 0.0
C
C     SOLVE THE EQUATIONS WITH DOUBLE PRECISION ROUTINE
      CALL DPMATS(ADP,MM,BDP,1,DETDP)
C
C     ADD THE CORRECTIONS TO THE SOLUTIONS
C
      DO 55 I = 1,MM
   55 XDP(I) = XDP(I) + BDP(I)
C
C     BACKSUBSTITUTE AND CALCULATE THE RESIDUALS
C
      DO 70 I = 1,MM
      DO 60 J = 1,MM
   60 BACKDP(I) = BACKDP(I) + XDP(J)*ADPS(I,J)
   70 ERRDP(I) = BDPS(I) - BACKDP(I)
C
C     PRINT THE RESULTS OF THIS ITERATION
C
      PRINT 999, ICOUNT
  999 FORMAT(1H1,11H ITERATION      , I4, /// )
      PRINT 992
  992 FORMAT(8X,'I',10X,'X NEW',17X,'RHS',16X,'BACK',13X,
     *'CORRECTION',14X,'ERROR',///)
      PRINT 991,(I,XDP(I),BDPS(I),BACKDP(I),BDP(I),ERRDP(I),I=1,MM)
  991 FORMAT(I10,5E20.8)
```

```
C
      DO 75 I = 1,MM
   75 BDP(I) = ERRDP(I)
C     END OF LOOP FOR ITERATION
C
 9990 CONTINUE
C
      DO 80 I = 1,MM
   80 X(I) = XDP(I)
C
      RETURN
      END
```

```
      SUBROUTINE DPMATS(A,N,B,M,DETERM)
C     F1 UCSD DPMATS69
C     DOUBLE PRECISION MATRIX INVERSION WITH ACCOMPANYING SOLUTION OF LINEAR
C     EQUATIONS
CDIMENSIONS FOR MATINV ARE IPIVOT(N),A(N,N),B(N,1),INDEX(N,2),PIVOT(N),
C  N IS THE MAXIMUM VALUE FOR N DEGRE.
      TYPE DOUBLE A,B,DETERM,AMAX,T,SWAP,PIVOT,DABS
      DIMENSION IPIVOT(27),A(27,27),B(27,1),INDEX(27,2),PIVOT(27)
      EQUIVALENCE (IROW,JROW), (ICOLUM,JCOLUM), (AMAX, T, SWAP)
C
C     INITIALIZATION
C
   10 DETERM=1.0
   15 DO 20 J=1,N
   20 IPIVOT(J)=0
   30 DO 550 I=1,N
C
C     SEARCH FOR PIVOT ELEMENT
C
   40 AMAX=0.0
   45 DO 105 J=1,N
   50 IF (IPIVOT(J)-1) 60, 105, 60
   60 DO 100 K=1,N
   70 IF (IPIVOT(K)-1) 80, 100, 740
   80 IF(DABS(AMAX).LT.DABS(A(J,K)))85,100
   85 IROW=J
   90 ICOLUM=K
   95 AMAX=A(J,K)
  100 CONTINUE
  105 CONTINUE
  110 IPIVOT(ICOLUM)=IPIVOT(ICOLUM)+1
C
C     INTERCHANGE ROWS TO PUT PIVOT ELEMENT ON DIAGONAL
C
  130 IF (IROW-ICOLUM) 140, 260, 140
  140 DETERM=-DETERM
  150 DO 200 L=1,N
  160 SWAP=A(IROW,L)
  170 A(IROW,L)=A(ICOLUM,L)
  200 A(ICOLUM,L)=SWAP
  205 IF(M) 260, 260, 210
  210 DO 250 L=1, M
  220 SWAP=B(IROW,L)
  230 B(IROW,L)=B(ICOLUM,L)
  250 B(ICOLUM,L)=SWAP
  260 INDEX(I,1)=IROW
  270 INDEX(I,2)=ICOLUM
  310 PIVOT(I)=A(ICOLUM,ICOLUM)
C
C
C 320 DETERM=DETERM*PIVOT(I)
C
C
C
C     DIVIDE PIVOT ROW BY PIVOT ELEMENT
C
  330 A(ICOLUM,ICOLUM)=1.0
  340 DO 350 L=1,N
  350 A(ICOLUM,L)=A(ICOLUM,L)/PIVOT(I)
  355 IF(M) 380, 380, 360
  360 DO 370 L=1,M
  370 B(ICOLUM,L)=B(ICOLUM,L)/PIVOT(I)
```

```
C
C     REDUCE NON-PIVOT ROWS
C
  380 DO 550 L1=1,N                                                     DPMTS 42
  390 IF(L1-ICOLUM) 400, 550, 400                                       DPMTS 43
  400 T=A(L1,ICOLUM)                                                    DPMTS 44
  420 A(L1,ICOLUM)=0.0                                                  DPMTS 45
  430 DO 450 L=1,N                                                      DPMTS 46
  450 A(L1,L)=A(L1,L)-A(ICOLUM,L)*T                                     DPMTS 47
  455 IF(M) 550, 550, 460                                               DPMTS 48
  460 DO 500 L=1,M                                                      DPMTS 49
  500 B(L1,L)=B(L1,L)-B(ICOLUM,L)*T                                     DPMTS 50
  550 CONTINUE                                                          DPMTS 51
C
C     INTERCHANGE COLUMNS
C
  600 DO 710 I=1,N                                                      DPMTS 52
  610 L=N+1-I                                                           DPMTS 53
  620 IF (INDEX(L,1)-INDEX(L,2)) 630, 710, 630                          DPMTS 54
  630 JROW=INDEX(L,1)                                                   DPMTS 55
  640 JCOLUM=INDEX(L,2)                                                 DPMTS 56
  650 DO 705 K=1,N                                                      DPMTS 57
  660 SWAP=A(K,JROW)                                                    DPMTS 58
  670 A(K,JROW)=A(K,JCOLUM)                                             DPMTS 59
  700 A(K,JCOLUM)=SWAP                                                  DPMTS 60
  705 CONTINUE                                                          DPMTS 61
  710 CONTINUE                                                          DPMTS 62
  740 RETURN                                                            DPMTS 63
  750 END                                                               DPMTS 64
```

```
      SUBROUTINE DOUBLE(X0,X1,Y0,Y1,TEST,LIM,VOLUME,INTX,R ,F)
C
C        ARGUMENTS ..
C
C        X0         LOWER LIMIT OF OUTER INTEGRAL (INPUT)
C        X1         UPPER LIMIT OF OUTER INTEGRAL (. PUT)
C        Y0         LOWER LIMIT OF INNER INTEGRAL (INPUT)
C        Y1         UPPER LIMIT OF INNER INTEGRAL (INPUT)
C        TEST       MAXIMUM TOLERABLE RELATIVE ERROR FOR OUTER INTEGRAL (INPUT)
C        LIM        MAXIMUM NUMBER OF SUBDIVISIONS FOR BOTH INTEGRALS (INPUT)
C        VOLUME     VALUE OF THE DOUBLE INTEGRAL (OUTPUT)
C        INTX       (2**INTX) = NUMBER OF SUBDIVISIONS FOR OUTER INTEGRAL (OUTPUT
C        R          RELATIVE ERROR FOR THE OUTER  INTEGRAL (OUTPUT)
C        F          NAME OF FUNCTION TO BE INTEGRATED (INPUT)
C     THE RELATIVE ERROR OF THE INNER INTEGRAL IS TEN TIMES SMALLER
C     THAN THAT FOR THE OUTER INTEGRAL
      NOIX=KOUNT=  0                                                    DOUBLE
      ODD = EVEN = VOLUM1 = 0.0
      INTX = V =   1.0                                                  DOUBLE
      R1  =     10.0                                                    DOUBLE
      TES   =  TEST / 10.
      CALL INNER(X0,Y0,Y1,TES,LIM,FACE0,NUMBR,ARE  ,F            )     DOUBLE
      CALL INNER(X1,Y0,Y1,TES,LIM,FACE1,NUMBR,ARE   ,F           )     DOUBLE
C     INNER IS SIMCON8 MODIFIED TO REFER TO A FUNCTION F(X,Y).
      FACES = FACE0 + FACE1
 2    DELTX = (X1 - X0)/V                                               DOUBLE
      ODD   =   EVEN + ODD
      X   =  X0 + DELTX/2.
      EVEN = 0.0
      DO 3 I = 1, INTX                                                  DOUBLE
      CALL INNER(X,Y0,Y1,TES,LIM,SECTN,NUMBR,ARE   ,F            )     DOUBLE
      EVEN   =  EVEN + SECTN
      X=  X + DELTX
 3    CONTINUE
      VOLUME=(FACES+4.0*EVEN+2.0*ODD)*DELTX/6.
      NOIX = NOIX + 1                                                   DOUBLE
      R= ABSF(1. - (VOLUM1/VOLUME))                                     DOUBLE
      IF(R.GE.R1) 5,31
 31   IF(NOIX      .GE.    LIM     )     35,     32                     DOUBLE
 32   IF(R.LE.TEST)   35,33
 33   VOLUM1 = VOLUME
      INTX = INTX* 2
      V = V*2.
      GO TO 2
 35   RETURN
 5    IF(KOUNT.GE.3)   55,51
 51   KOUNT = KOUNT + 1
      R1 = R
      GO TO 2
 55   PRINT 56, VOLUME, R, NOIX                                         DOUBLE
 56   FORMAT(30H OUTER INTEGRAL NOT CONVERGING , 2F15.6,I6           )  DOUBLE
      RETURN                                                            DOUBLE
      END                                                               DOUBLE
```

```
      FUNCTION F (THETA,R)
      ENTRY F1
      CALL UR1(U1,R,THETA)
      CALL UTH1(U2,R,THETA)
      F =(U1**2 +U2**2)*R
      RETURN
C
C
      ENTRY F2
      CALL UR2(U1,R,THETA)
      CALL UTH2(U2,R,THETA)
      F =(U1**2+U2**2)*R
      RETURN
      END
```

```
      SUBROUTINE INNER(ABSCIS,X1,XEND,TEST,LIM,AREA,NOI,R,F)
C     DI UCSD SIMCON, REVISED MARCH 1967 TO REFER TO 2-ARGUMENT FUNCTIONS
C     AND TEST THEIR CONVERGENCE.
      NOI = KOUNT = 0
      R1 = 10.0
      ODD=0.0
      INT=1
      V=1.0
      EVEN=0.0
      AREA1=0.0
 19   ENDS= F(ABSCIS,X1) + F(ABSCIS,XEND)
 2    H=(XEND-X1)/V                                                    INNER
      ODD=EVEN+ODD
      X=X1+H/2.
      EVEN=0.0
      DO 3 I=1,INT
 21   EVEN=EVEN+ F(ABSCIS,X)
      X=X+H
 3    CONTINUE
 31   AREA=(ENDS      +4.0*EVEN+2.0*ODD)*H/6.0
      NOI=NOI+1
 34   R=ABSF((AREA1-AREA)/AREA        )                                INNER
      IF(R.GE.R1)  50, 3405
 3405 R1=R
      IF(NOI-LIM) 341,60,60                                            INNER
 341  IF(R-TEST) 35,35,4                                               INNER
 35   RETURN
 4    AREA1=AREA
 46   INT=2*INT
      V=2.0*V
      GO TO 2
 50   IF(KOUNT.GE.3) 55,51
 51   KOUNT = KOUNT + 1
      R1=R
      GO TO 2
 55   PRINT 56, AREA, R, NOI, ABSCIS                                   INNER
 56   FORMAT( 30H INNER INTEGRAL NOT CONVERGING , 2F15.6,I6,F15.6     ) INNER
      RETURN
 60   PRINT 61                                                         INNER
 61   FORMAT (42H USED UP SPLITTINGS. RETURNING TO DOUBLE.            ) INNER
      RETURN                                                           INNER
      END
```

```
      SUBROUTINE LISTARAY (A, NMAX, M, N, ISTART)
C
      DIMENSION A(NMAX, 1)
      DIMENSION IC(10), IFORM(10)                                       LARY   3
      DATA (  IFORM  =  8H(1I)     ,                                    LARY   4
     *                  8H(/,5H6RO,                                     LARY   5
     *                  8H         ,                                    LARY   6
     *                  8HM  COL,I,                                     LARY   7
     *                  8H4,2X),5H,                                     LARY   8
     *                  8H  ROW)   ,                                    LARY   9
     *                  8H(I4,1X,  ,                                    LARY  10
     *                  8H         ,                                    LARY  11
     *                  8HI5)      ,                                    LARY  12
     *                  8H(1H5)       )                                 LARY  13
C
    2 IFORM(3) =        8HW   ,10I5                                     LARY  14
      IFORM(8) =        8H10E11.3,                                      LARY  15
      IPAGE = JPAGE = 1                                                 LARY 115
      IF(M .LE. 25) JPAGE = 2                                           LARY 215
      DO 10 I=ISTART, N, 10                                             LARY  16
      IM1 = I-1      $      IP9 = I+9      $      LAST = 10             LARY  17
      IF(IP9 .GT. N) 6,7                                                LARY  18
    6 IP9 = N      $      J = N-ISTART+1      $      LAST = J-(10*(J/10)) LARY  19
      ENCODE(8, 900, IFORM(3)) LAST                                     LARY  20
      ENCODE(8, 901, IFORM(8)) LAST                                     LARY  21
  900 FORMAT (4HW   ,,I2,2HI5)                                          LARY  22
  901 FORMAT (I2,6HE11.3,)                                              LARY  23
    7 DO 8 ICOUNT = 1, LAST                                             LARY  24
    8 IC(ICOUNT) = IM1 + ICOUNT                                         LARY  25
      PRINT IFORM(1),IPAGE                                              LARY  26
      PRINT IFORM(2), (IC(J), J=1,LAST)                                 LARY  27
      PRINT IFORM(7), (J, (A( J,II),II=I, IP9), J, J=1,M)               LARY  28
      PRINT IFORM(2), (IC(J), J=1,LAST)                                 LARY  29
   10 CONTINUE                                                          LARY  30
      PRINT IFORM(10)
      RETURN      $      END                                            LARY  32
```

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13. ABSTRACT

Part I of this report covers the problem of free and forced vibration of a unidirectional, multifiber reinforced composite. A theoretical investigation is conducted through the use of the linear theory of elasticity. For this case, the geometrical array of the fiber representative element consists of a circular, inner solid fiber cylinder bounded by and bonded to a circular outer matrix shell. Composites of infinite, finite, and semi-infinite lengths are treated. It is assumed that the deformation is axisymmetrical and that the vibration is longitudinal. Characteristic equations are established which relate circular frequencies to axial wave numbers for three cases of composite length. Solutions are obtained for stresses and displacements of composites, of finite or semi-infinite length, subjected to axial, piecewise-constant, or sinusoidal loading at one end and different geometrical boundary conditions at the other. Part II presents an approximate differential equation based on the Bernoulli hypothesis of deformation. The solution of this equation is established for steady and transient states of vibration in composites of both finite and infinite length. Computation of the coefficients in the differential equation is performed by assuming symmetry of revolution for the basic element and also by using a hexagonal fiber arrangement. Part III lists numerical results based on the equations developed in Parts I and II. The appendixes to this report give the computer programs used to perform the computations.

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